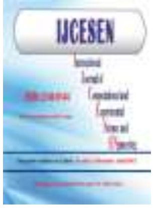
International Journal of Computational and Experimental Science and ENgineering
(IJCESEN)

Vol. 12-No.2 (2026) pp. 368-373
http://www.ijcesen.com
ISSN: 2149-9144

Research Article



Filter Regularization Method for an Inverse Problem of Over-damped Harmonic Oscillator Equation

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Article Info:

DOI: 10.22399/ijcesen.5212
Received : 20 November 2025
Revised : 03 April 2026
Accepted : 04 May 2026

Keywords

Filter regularization method
Harmonic oscillator equation
ill posed problem
inverse problem

Abstract:

In many fields of applied mathematics and physics, inverse problems involving the harmonic oscillator are common. These problems are often ill-posed in the sense of Jacques Hadamard, where minor changes in the data can result in significant deviations in the solution. In this work, we study an ill-posed inverse problem related to the harmonic oscillator equation with the aim of reconstructing unknown parameters from noisy and indirect observations.We suggest a technique of regularization to address the instability present in this class of problems. Convergence results are obtained under appropriate assumptions on the regularization parameter and noise level, and the regularized problem's well posed-ness is established. Furthermore, error estimates are obtained to measure how stable the suggested method is. The efficiency and resilience of the approach in recovering stable approximations of the intended solution are illustrated through numerical simulations. The obtained results verify that the suggested regularization technique provides a dependable and effective method for resolving harmonic oscillator inverse problems.

1. Introduction

A harmonic oscillator is a fundamental system in physics and mathematics. An oscillator that loses energy due to friction, resistance or any kind of damping force is known as a damped harmonic oscillator. it models many real world systems like a car suspension, a pendulum in air, or an RLC electrical circuit.

Applying Newton's second law that force equals mass times acceleration, we discover that when a damping force, like friction or air resistance, opposes the motion and the spring pulls the mass back toward equilibrium, a damped harmonic oscillator equation is created. The strength of damping in relation to the restoring force determines how the system acts for that there exist three types of damping : Under-damped (the damping rate less than one: oscillations with shrinking amplitude), critically damped (the damping rate equal to one: Smooth return no oscillation), over-damped (the damping rate greater than one: slow return, no oscillation) see [1][2][3]. Mckean H.P and Trubowitz E [11] studied the equation of harmonic oscillator on R. They gave an algorithm to reconstruct the potential from special constants derived from eigenfunctions. Levitan [10] reproved some results of Mckean H.P and Trubowitz E [11]. After some mathematicians such as Gesztesy, Simon [7], Chelkak [1], Kargaev [3], Korotyaev [4] proved existence and uniqueness. We result that earlier works gave algorithms and partial results while later works proved uniqueness and characterization.

Let y: [0, T] → H and H is Hilbert space.

In H Hilbert space, consider the following problem:

$$\begin{cases} \dfrac{d^2y(t)}{dt^2} + 2\zeta\omega\dfrac{dy(t)}{dt} + \omega^2 = 0 & t \in [0;T] \quad (1) \\ y(T) = f & (2) \\ y'(T) = g & (3) \end{cases}$$

Where $(f; g) \in H \times H$

Note that ζ: the damping rate, ω: the natural frequency such as: $\omega = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2\sqrt{km}}$

Let y: [0, T] → H is a solution of the problem if:

1) y(t) is twice derivable then y(t) ∈ C2 (0, T ; H).
2) For t∈ [0, T]; y (t) satisfy the equation (1) and the conditions (2)-(3).

The general solution of the equation (1) is given by: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. Where r is a solution of the corresponding characteristic equation.

Using conditions (2)-(3) we obtain:

$$\begin{cases} c_1 = \dfrac{r_2 f - g}{(r_2 - r_1) e^{r_1 T}} \\ c_2 = \dfrac{g - r_1 f}{(r_2 - r_1) e^{r_2 T}} \end{cases}$$

With

$$\begin{cases} y(T) = f \text{ then } c_1 e^{r_1 T} + c_2 e^{r_2 T} = f \\ y'(T) = g \text{ then } c_1 r_1 e^{r_1 T} + c_2 r_2 e^{r_2 T} = g \end{cases}$$

We get: $y(t) = \frac{r_2 f - g}{(r_2 - r_1) e^{r_1 T}} e^{r_1 t} + \frac{g - r_1 f}{(r_2 - r_1) e^{r_2 T}} e^{r_2 t} \quad (4)$

Therefore, if the solution of the problem exist it has this formula (4).

Above we check the ill posed-ness of this problem, where generally the solution is unbounded. As consequence, regularization techniques are necessary to obtain well-posed problem. We propose a regularization utilizing a filter method. This method is based on adding a filter function that satisfies specific conditions, allowing us to obtain a general solution for well-posed problem.

Consider the problem with noisy data:

$$\begin{cases} \tilde{y}''(t) + 2\zeta\omega\tilde{y}'(t) + \omega^2 \tilde{y}(t) = 0 \qquad t \in [0; T] \\ \tilde{y}(T) = \tilde{f} \\ \tilde{y}'(T) = \tilde{g} \end{cases}$$

Where $\tilde{f}, \tilde{g} \in H$ and $|\tilde{f} - f| \le \delta$, $|\tilde{g} - g| \le \delta$

$$\|\tilde{y}(t) - y(t)\|_H = \left\| \frac{r_2\tilde{f} - \tilde{g}}{(r_2 - r_1) e^{r_1 T}} e^{r_1 t} + \frac{\tilde{g} - r_1\tilde{f}}{(r_2 - r_1) e^{r_2 T}} e^{r_2 t} - \left(\frac{r_2 f - g}{(r_2 - r_1) e^{r_1 T}} e^{r_1 t} + \frac{g - r_1 f}{(r_2 - r_1) e^{r_2 T}} e^{r_2 t} \right) \right\|$$

$$= \left\| \begin{matrix} \dfrac{r_2(\tilde{f} - f) - (\tilde{g} - g)}{(r_2 - r_1) e^{r_1 T}} e^{r_1 t} \\ + \dfrac{(\tilde{g} - g) - r_1(\tilde{f} - f)}{(r_2 - r_1) e^{r_2 T}} e^{r_2 t} \end{matrix} \right\|$$

$$\le \left\| \begin{matrix} \dfrac{r_2(\tilde{f} - f) - (\tilde{g} - g)}{(r_2 - r_1) e^{r_1 T}} \\ + \dfrac{(\tilde{g} - g) - r_1(\tilde{f} - f)}{(r_2 - r_1) e^{r_2 T}} \end{matrix} \right\|$$

$$\le \left\| \begin{matrix} \dfrac{r_2(\tilde{f} - f) - (\tilde{g} - g)}{r_2 - r_1} e^{|r_1| T} \\ + \dfrac{(\tilde{g} - g) - r_1(\tilde{f} - f)}{r_2 - r_1} e^{|r_2| T} \end{matrix} \right\|$$

We pose $r = \max\{|r_1|; |r_2|\}$

$$\|\tilde{y}(t) - y(t)\|_H \le \left\| \begin{matrix} \dfrac{r_2(\tilde{f} - f) - (\tilde{g} - g)}{r_2 - r_1} e^{rT} \\ + \dfrac{(\tilde{g} - g) - r_1(\tilde{f} - f)}{r_2 - r_1} e^{rT} \end{matrix} \right\|$$

$$\le \|(\tilde{f} - f) e^{rT}\|$$

$$\le \delta e^{rT} \to +\infty \text{ when } T \to +\infty$$

We note that e^{rT} effects the stability of the solution.

2 Filter regularization method

Let measured information $y^{\delta}(T) \in H$ and y(T) ∈ H satisfy $\|y^{\delta}(T) - y(T)\|_H \le \varepsilon$

We consider a function q(α, r) called the filter function and α is a parameter of regularization We define a regularized solution by:

$$y_\alpha^\delta(t) = q(\alpha, r) \left(\begin{matrix} \dfrac{r_2\tilde{f} - \tilde{g}}{(r_2 - r_1) e^{r_1 T}} e^{r_1 t} \\ + \dfrac{\tilde{g} - r_1\tilde{f}}{(r_2 - r_1) e^{r_2 T}} e^{r_2 t} \end{matrix} \right)$$

Theorem 2.1.

let y(t) a exact solution of problem(1)-(3) and $y_\alpha^\delta(t)$ regularized solution such as $\|y^{\delta}(T) - y(T)\|_H \le \varepsilon$ and $\|y^{\delta\prime}(T) - y'(t)\|_H \le \varepsilon$ with $\left\|\frac{dy}{dt}(0)\right\|_H \le E_1$ and $\left\|\frac{dy'}{dt}(0)\right\|_H \le E_2$

And q(α, r) the filter function satisfy these conditions:

1. $\sup_r |q(\alpha, r)| \le C_2(\alpha)$
2. $\sup_r |q(\alpha, r) e^{rT}| \le k_1(\alpha)$
3. $\sup_r |q(\alpha, r) - 1| \le k_2(\alpha)$
4. $\sup_r |q(\alpha, r) - 1| e^{-rT} \le C_1(\alpha)$

We suppose that the parameter of regularization α = α (δ) satisfies the following conditions:

a. $\delta C_2(\alpha) \underset{\delta \to 0}{\to} 0$
b. $C_1(\alpha) \underset{\delta \to 0}{\to} 0$
c. $\delta k_1(\alpha) \underset{\delta \to 0}{\to} 0$
d. $k_2(\alpha) \underset{\delta \to 0}{\to} 0$

Proof

For every $0 \le t < T$

$$\left\|y_\alpha^\delta(t)-y(t)\right\|_H=\left\|\begin{array}{c} q(\alpha,r)\left(\begin{array}{c}\frac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +\frac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\\ -\left(\begin{array}{c}\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\end{array}\right\|$$

$$\leq\left\|\begin{array}{l} q(\alpha,r)\left(\begin{array}{c}\frac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +\frac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\\ -q(\alpha,r)\left(\begin{array}{c}\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\end{array}\right\|$$

$$+\left\|\begin{array}{c} q(\alpha,r)\left(\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}+\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}\right)\\ -\left(\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}+\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}\right)\end{array}\right\|$$

$$\left\|y_\alpha^\delta(t)-y(t)\right\|_H\leq\left\{\begin{array}{c}\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)e^{rT}\right|\left\|\tilde{f}-f\right\|\\ +\\ \underset{r}{\mathrm{Sup}}\left|q(\alpha,r)-1\right|\left\|fe^{rT}\right\|\end{array}\right.$$

$$\leq\delta\,\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)e^{rT}\right|+E_1\,\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)-1\right|$$

$$\leq\delta k_1(\alpha)+E_1k_2(\alpha)$$

For t=T

$$\left\|y_\alpha^\delta(T)-y(T)\right\|_H=\left\|\begin{array}{c} q(\alpha,r)\left(\frac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)}+\frac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)}\right)\\ -\left(\frac{r_2f-g}{(r_2-r_1)}+\frac{g-r_1f}{(r_2-r_1)}\right)\end{array}\right\|$$

$$\leq\left\|\begin{array}{l} q(\alpha,r)\left(\frac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)}+\frac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)}\right)\\ -q(\alpha,r)\left(\frac{r_2f-g}{(r_2-r_1)}+\frac{g-r_1f}{(r_2-r_1)}\right)\end{array}\right\|$$

$$+\left\|\begin{array}{c} q(\alpha,r)\left(\frac{r_2f-g}{(r_2-r_1)}+\frac{g-r_1f}{(r_2-r_1)}\right)\\ -\left(\frac{r_2f-g}{(r_2-r_1)}+\frac{g-r_1f}{(r_2-r_1)}\right)\end{array}\right\|$$

$$\left\|y_\alpha^\delta(T)-y(T)\right\|_H\leq\left\{\begin{array}{c}\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)\right|\left\|\tilde{f}-f\right\|\\ +\underset{r}{\mathrm{Sup}}\left|(q(\alpha,r)-1)e^{-rT}\right|\left\|fe^{rT}\right\|\end{array}\right.$$

$$\leq\delta\,\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)\right|+E_1\,\underset{r}{\mathrm{Sup}}\left|(q(\alpha,r)-1)e^{-rT}\right|$$

$$\leq\delta C_2(\alpha)+E_1C_1(\alpha)$$

For every $0<t<T$

$$\left\|y_\alpha^{\delta\prime}(t)-y'(t)\right\|_H=\left\|\begin{array}{c} q(\alpha,r)\left(\begin{array}{c}r_1\frac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +r_2\frac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\\ -\left(\begin{array}{c}r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\end{array}\right\|$$

$$\leq\left\|\begin{array}{l} q(\alpha,r)\left(\begin{array}{c}r_1\frac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +r_2\frac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\\ -q(\alpha,r)\left(\begin{array}{c}r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\end{array}\right\|$$

$$+\left\|\begin{array}{c} q(\alpha,r)\left(\begin{array}{c}r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\\ -\left(\begin{array}{c}r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}\\ +r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}\end{array}\right)\end{array}\right\|$$

$$\left\|y_\alpha^{\delta\prime}(t)-y'(t)\right\|_H\leq\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)\right|\left\|\tilde{g}-g\right\|+\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)-1\right|\left\|ge^{rT}\right\|$$

$$\leq\delta\,\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)\right|+E_2\,\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)-1\right|$$

$$\leq\delta C_2(\alpha)+E_2k_2(\alpha)$$

For t=0

$$\left\|y_\alpha^{\delta\prime}(0)-y'(0)\right\|_H=\left\|\begin{array}{c} q(\alpha,r)\left(r_1\frac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)e^{r_1T}}+r_2\frac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)e^{r_2T}}\right)\\ -\left(r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}+r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}\right)\end{array}\right\|$$

$$\leq\left\|\begin{array}{l} q(\alpha,r)\left(r_1\frac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)e^{r_1T}}+r_2\frac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)e^{r_2T}}\right)\\ -q(\alpha,r)\left(r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}+r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}\right)\end{array}\right\|$$

$$+\left\|\begin{array}{c} q(\alpha,r)\left(r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}+r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}\right)\\ -\left(r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}+r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}\right)\end{array}\right\|$$

$$\left\|y_\alpha^{\delta\prime}(0)-y'(0)\right\|_H\leq\left\{\begin{array}{c}\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)e^{rT}\right|\left\|\tilde{g}-g\right\|\\ +\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)-1\right|\left\|ge^{rT}\right\|\end{array}\right.$$

$$\leq\delta\,\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)e^{rT}\right|+E_2\,\underset{r}{\mathrm{Sup}}\left|q(\alpha,r)-1\right|$$

$$\leq\delta k_1(\alpha)+E_2k_2(\alpha)$$

For t=T

$$\left\|y_\alpha^{\delta\prime}(T)-y'(T)\right\|_H=\left\|\begin{matrix} q(\alpha,r)\left(r_1\dfrac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)}+r_2\dfrac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)}\right) \\ -\left(r_1\dfrac{r_2f-g}{(r_2-r_1)}+r_2\dfrac{g-r_1f}{(r_2-r_1)}\right)\end{matrix}\right\|$$

$$\leq\left\|\begin{matrix} q(\alpha,r)\left(r_1\dfrac{r_2\tilde{f}-\tilde{g}}{(r_2-r_1)}+r_2\dfrac{\tilde{g}-r_1\tilde{f}}{(r_2-r_1)}\right) \\ -q(\alpha,r)\left(r_1\dfrac{r_2f-g}{(r_2-r_1)}+r_2\dfrac{g-r_1f}{(r_2-r_1)}\right)\end{matrix}\right\|$$

$$+\left\|\begin{matrix} q(\alpha,r)\left(r_1\dfrac{r_2f-g}{(r_2-r_1)}+r_2\dfrac{g-r_1f}{(r_2-r_1)}\right) \\ -\left(r_1\dfrac{r_2f-g}{(r_2-r_1)}+r_2\dfrac{g-r_1f}{(r_2-r_1)}\right)\end{matrix}\right\|$$

Then

$$\left\|y_\alpha^{\delta\prime}(T)-y'(T)\right\|_H\leq\left\{\begin{matrix}\sup_r|q(\alpha,r)|\|\tilde{g}-g\| \\ +\sup_r|q((\alpha,r)-1)e^{-rT}|\|ge^{rT}\|\end{matrix}\right.$$

$$\leq\delta\sup_r|q(\alpha,r)|+E_2\sup_r|(q(\alpha,r)-1)e^{-rT}|$$

$$\leq\delta C_2(\alpha)+E_2C_1(\alpha)$$

According to the theorem (2.1) we have: for every $0\leq t<T$

$$\left\|y_\alpha^\delta(t)-y(t)\right\|_H\leq\delta k_1(\alpha)+E_1k_2(\alpha)$$

$$\left\|y_\alpha^{\delta\prime}(t)-y'(t)\right\|_H\leq\delta C_2(\alpha)+E_2k_2(\alpha)$$

3 Applications

We have : $y(t)=\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}+\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}$

Then : $y'(t)=r_1\frac{r_2f-g}{(r_2-r_1)e^{r_1T}}e^{r_1t}+r_2\frac{g-r_1f}{(r_2-r_1)e^{r_2T}}e^{r_2t}$

If $g=e^{\frac{-1}{2}rT}$ we have :

$$y'(0)=r_1\frac{r_2f-e^{\frac{-1}{2}rT}}{(r_2-r_1)e^{r_1T}}+r_2\frac{e^{\frac{-1}{2}rT}-r_1f}{(r_2-r_1)e^{r_2T}}$$

$$\|y'(0)\|^2=\left\|r_1\frac{r_2f-e^{\frac{-1}{2}rT}}{(r_2-r_1)e^{r_1T}}+r_2\frac{e^{\frac{-1}{2}rT}-r_1f}{(r_2-r_1)e^{r_2T}}\right\|$$

$$\leq\left\|r_1\frac{r_2f-e^{\frac{-1}{2}rT}}{(r_2-r_1)}e^{rT}+r_2\frac{e^{\frac{-1}{2}rT}-r_1f}{r_2-r_1}e^{rT}\right\|$$

$$\leq e^{rT}\to+\infty \text{ when } t\to+\infty$$

So our goal is to limit the term e^{rT} .

We put some examples proposed in other problems, but those verify our conditions about the filter function.

Example 1: The function $q(\alpha,r)=\frac{1}{1+\alpha e^{rT}}$ is a filter function. In effect, this function verifies all conditions of filter function. It proposed in [8] for a backward heat conduction problem

1. $|q(\alpha,r)|=\left|\frac{1}{1+\alpha e^{rT}}\right|\leq1=C_2(\alpha)$
2. $|q(\alpha,r)e^{rT}|=\left|\frac{e^{rT}}{1+\alpha e^{rT}}\right|\leq\frac{1}{\alpha}=k_1(\alpha)$
3. $|q(\alpha,r)-1|=\left|\frac{\alpha e^{rT}}{1+\alpha e^{rT}}\right|\leq\left|\frac{1}{1+\frac{1}{\alpha}e^{-rT}}\right|\leq\alpha=k_2(\alpha)$
4. $|q(\alpha,r)-1|e^{-rT}=\left|\frac{-\alpha}{1+\alpha e^{rT}}\right|\leq\alpha=C_1(\alpha)$

Example 2: The function $q(\alpha,r,s)=\frac{1}{1+\alpha^se^{rT}}$; $s>1$ is also filter function and satisfy all conditions. Hai-Hua Oin and Ting Wei suggested this function in [8]

1. $|q(\alpha,r,s)|=|\frac{1}{1+\alpha^se^{rT}}|\leq1=C_2(\alpha)$
2. $|q(\alpha,r,s)e^{rT}|=\left|\frac{e^{rT}}{1+\alpha^se^{rT}}\right|\leq\frac{1}{\alpha^s}=k_1(\alpha)$
3. $|q(\alpha,r,s)-1|=\left|\frac{\alpha^se^{rT}}{1+\alpha^se^{rT}}\right|\leq\left|\frac{1}{1+\frac{1}{\alpha^s}e^{-rT}}\right|\leq\alpha^s=k_2(\alpha)$
4. $|q(\alpha,r,s)-1|e^{-rT}=\left|\frac{-\alpha^s}{1+\alpha^se^{rT}}\right|\leq\alpha^s=C_1(\alpha)$

Example 3: Recently, Kirane M, Tuan N H, Luu V innovated the function $q(\alpha,r)=e^{-\alpha e^{rT}}$ in [9] for an inverse parabolic problem in several variables. Indeed, we check this function is a filter function.

1. $|q(\alpha,r)|=\left|e^{-\alpha e^{rT}}\right|\leq1=C_2(\alpha)$
2. $|q(\alpha,r)e^{rT}|=\left|e^{rT-\alpha e^{rT}}\right|\leq1=k_1(\alpha)$
3. $|q(\alpha,r)-1|=\left|e^{-\alpha e^{rT}}-1\right|\leq\alpha e^{rT}=k_2(\alpha)$
4. $|q(\alpha,r)-1|e^{-rT}=\left|e^{-rT-\alpha e^{rT}}-e^{-rT}\right|\leq\alpha=C_1(\alpha)$

Example 4: The Algerians mathematics Denche Mohamed, Bessila K in [6] found the function $q(\alpha,r)=\frac{e^{-rT}}{\alpha+e^{-rT}}$ which is a filter function satisfying all conditions.

1. $|q(\alpha,r)|=\left|\frac{e^{-rT}}{\alpha+e^{-rT}}\right|\leq1=C_2(\alpha)$
2. $|q(\alpha,r)e^{rT}|=\left|\frac{1}{\alpha+e^{-rT}}\right|\leq\frac{1}{\alpha}=k_1(\alpha)$
3. $|q(\alpha,r)-1|=\left|\frac{\alpha}{\alpha+e^{-rT}}\right|\leq\left|\frac{1}{1+\frac{1}{\alpha}e^{rT}}\right|\leq\alpha=k_2(\alpha)$
4. $|q(\alpha,r)-1|e^{-rT}=\left|\frac{-\alpha e^{-rT}}{\alpha+e^{-rT}}\right|\leq\left|\frac{1}{\frac{1}{\alpha}e^{rT}+1}\right|\leq\alpha=C_1(\alpha)$

Example 5: The improved function

$q(\alpha,r,s)=\frac{e^{-rT}}{\alpha^s+e^{-rT}},S>1$ from the previous function in non-homogeneous backward Cauchy problem [5]. It is a filter function too.

1. $|q(\alpha,r,s)|=\left|\frac{e^{-rT}}{\alpha^s+e^{-rT}}\right|\leq1=C_2(\alpha)$
2. $|q(\alpha,r,s)e^{rT}|=\left|\frac{1}{\alpha^s+e^{-rT}}\right|\leq\frac{1}{\alpha^s}=k_1(\alpha)$
3. $|q(\alpha,r,s)-1|=\left|\frac{\alpha^s}{\alpha^s+e^{-rT}}\right|\leq\left|\frac{1}{1+\frac{1}{\alpha^s}e^{-rT}}\right|\leq\alpha^s=k_2(\alpha)$
4. $|q(\alpha,r,s)-1|e^{-rT}=\left|\frac{-\alpha^se^{-rT}}{1+\alpha e^{-rT}}\right|\leq\left|\frac{1}{\frac{1}{\alpha^s}+e^{rT}}\right|\leq\alpha^s=C_1(\alpha)$

4 Numerical example

In this section, we consider the simple example :

$$\begin{cases}\dfrac{d^2y}{dt^2}(t)+8\dfrac{dy}{dt}(t)+4y(t)=0\ ,t\in[0;2] \\ y(2)=e^{-1} \\ y'(2)=e^{-0.5}\end{cases}$$

We remark that T = 2, ζ = 2 and ω = 2 then the exact solution is:

$$y(t) = \frac{\left(-4+2\sqrt{3}\right)e^{-0.5} - e^{-1}}{\left(4\sqrt{3}\right)e^{-8-4\sqrt{3}}} e^{\left(-4-2\sqrt{3}\right)t} + \frac{e^{-1} - \left(-4-2\sqrt{3}\right)e^{-0.5}}{\left(4\sqrt{3}\right)e^{-8+4\sqrt{3}}} e^{\left(-4+2\sqrt{3}\right)t}$$

Let $q_1(t)$ the function filter of example 1 and $q_2(t)$the function filter of example 2 ...in the same way for the rest.

First, we fixed the parameter of regularization α and varied t then we compare the error and which is the best.

Table 1. we pose α = δ = 10^{-7}

t	*y(t)*	*q1(t)*	*q2(t)*	*q3(t)*	*q4(t)*	*q5(t)*
0	*−304294.*	*8984.9*	*9.25* × 10^{-5}	*9118.86*	*8984.92*	*9.25* × 10^{-5}
0.5	*−7284.41*	*215.08*	*2.21* × 10^{-6}	*218.29*	*215.08*	*2.21* × 10^{-6}
1	*−173.24*	*5.11*	*5.27* × 10^{-8}	*5.19*	*5.11*	*5.27* × 10^{-8}
1.5	*−3.25*	*0.09*	*9.89* × 10^{-10}	*0.09*	*0.09*	*9.89* × 10^{-10}
2	*0.6*	*0.01*	*1.84* × 10^{-10}	*0.01*	*0.01*	*1.84* × 10^{-10}

We can be observed that the error between the exact solution and the regularized solutions for different values of the parameter t. The result show that the error remains small and the functions in example 2 and 5 those functions have the same error, making them the best functions filters for this problem. Example 3 contains the worst function filter.

Table 2. we considered a smaller value of α = δ = 10^{-10}

t	*y(t)*	*q1(t)*	*q2(t)*	*q3(t)*	*q4(t)*	*q5(t)*
0	*-304294.9*	*9.25*	*5.82*× 10^{-11}	*9.25*	*9.25*	*1.16*× 10^{-10}
0.5	*-7284.4*	*0.22*	*1.81* × 10^{-12}	*0.22*	*0.22*	*2.72* × 10^{-12}
1	*-173.24*	*5.27* × 10^{-3}	*2.84* × 10^{-14}	*5.27* × 10^{-3}	*5.27* × 10^{-3}	*5.68* × 10^{-14}
1.5	*-3.25*	*9.89* × 10^{-5}	*8.88* × 10^{-16}	*9.89* 10^{-5}	*9.89* 10^{-5}	*8.88* × 10^{-16}
2	*0.60*	*1.84* × 10^{-5}	*1.11* × 10^{-16}	*1.84* × 10^{-5}	*1.84* × 10^{-5}	*2.22* × 10^{-16}

We remark the same behavior as in the table 1. But, in this case, the error was even smaller compared to the table 1 and the best functions filter for this problem is that of the example 2. Those functions have the same error. The worst function filter is the function of example 1 and 4. We result that the error vanishes for α near 0.

4. Conclusions

In this work, we investigated an ill posed inverse problem associated with the harmonic oscillator equation. Due to the instability of such problems in the sense of Hadamard, regularization techniques are required to obtain stable approximations of the solution. Several regularization methods were applied and compared in order to evaluate their effectiveness in stabilizing the inverse problem.The comparative analysis shows that each method provides a stable reconstruction of the solution, but their performance differs in terms of stability, accuracy and sensitivity to noise data.

Author Statements:

- **Ethical approval:** The conducted research is not related to either human or animal use.
- **Conflict of interest:** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper
- **Acknowledgement:** The authors declare that they have nobody or no-company to acknowledge.
- **Author contributions:** The authors declare that they have equal right on this paper.
- **Funding information:** The authors declare that there is no funding to be acknowledged.
- **Data availability statement:** The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.
- **Use of AI Tools:** The author(s) declare that no generative AI or AI-assisted technologies were used in the writing process of this manuscript.

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