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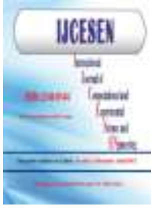

# The method of quasilinearization for scalar caputo fractional differential equation

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**Abstract:**

Non-linear problems occurring naturally in many branches of science, engineering, etc. are differential equations with integer order derivative. Fractional differential equations (FDEs) may model more effectively certain problems, than ordinary differential equations. Various methods for solving FDEs are being developed and these solutions have great importance in related fields. The method of replacing the original non-linear Caputo fractional differential equation by differential inequalities, whose associated equations can be solved, is known as 'quasilinearization'. In this paper we establish the method of upper and lower solutions coupled with the method of quasilinearization that converge to the solution of the Caputo fractional differential equation.

## 1. Introduction

Fractional Differential Equations (FDEs) model nonlinear problems arising in many branches of science, engineering, etc., in better way compared with its counterpart of integer derivatives. Some of the applications of Fractional differential and integral equations have been discussed in [1, 2, 3]. The use of fractional operator gives the freedom to choose the order of derivative; further the fractional operators are global in nature. This concept of differentiation and integration of arbitrary order is as old as the calculus itself. See [2, 3, 15].There are different definitions, which do not coincide in general as authors try to preserve different properties of the integer order derivatives and integrals. There are two main approaches to the fractional operators: the continuous and the discrete approach. The continuous approach is based on the Riemann-Liouville fractional integral and the discrete approach based on the Grunwald-Letnikov fractional derivative, defined as a limit of a fractional finite difference quotient. However, there exists a relation between the Riemann-Liouville and the Grunwald-Letnikov approach See [3,5]. The Riemann-Liouville definition played an important role in the development of the theory of fractional derivatives and integrals. However, in modelling of the problems in various disciplines, it was very difficult to interpret the initial conditions for the initial value problems. This makes Caputo fractional derivative and operator more suitable, these are closer to integer derivative results. See [2, 3,]. The Caputo derivative is defined and exists for most of the functions for which Riemann-Liouville derivative exists. See [5]. The applications of fractional dynamical systems have taken a dominant role in the past 40 years. Various methods for solving FDEs are being developed and these solutions have great importance in related fields. However, analytical or numerical computations of the solution of FDEs are challenging. This is mainly because some of the basic properties enjoyed by integer derivatives are not available. The explicit analytical solution in terms of familiar analysis is not possible. The works of [4, 5] provide analytical and numerical results related to FDEs. There exist a number of powerful procedures for obtaining approximate solutions of nonlinear FDEs See [6, 7, 8]. Various fixed point theorems are employed to establish qualitative properties, particularly the existence and uniqueness of solutions, for fractional differential equations. These methods in general are not usually computational methods and do not determine the interval of existence. Chapligin developed a method for obtaining approximate solutions of nonlinear differential equation with integer order derivative. See [11, 13]. The basic idea is to replace nonlinear differential equation by simpler differential equations and solutions of the simpler differential equations can be employed to 'bound' the solution of the original equation. This technique brings into play the theories of differential and integral inequalities and monotone operators. A systematic way of obtaining these bounding functions is furnished by the method known as 'quasilinearization'. The method of upper and lower solution combined with monotone method is an efficient tool to compute the minimal and maximal solution of FDEs. This method not only gives the existence of the solution, it also provides the interval of existence. See [6, 7, 8, 11, 12]. The advantage of this method for non-linear FDEs is sequences or iterates constructed does not require Mittag-Leffler function, which removes the computational complexity and approximations constructed converges to solution of FDEs. There are four different kinds of problems.

In this work, we provide a methodology to compute coupled lower and upper solutions of Type III Caputo fractional differential equation with initial conditions on any given interval.

## 2. Preliminary results

We use the following definitions and known results in our main results.

**Definition 2.1** Caputo fractional derivative of order '$q$' is given by the equation

$$ {}^cD^q\, x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q}\, x'(s)ds $$

where $0 < q < 1$, and $\Gamma(q)$ is the Gamma function.

Consider the Caputo fractional differential equation given by:

$$ {}^cD^q\, x = f(t,x) + g(t,x),\ x(t_0) = x_0 \qquad (1) $$

The corresponding Volterra fractional integral equation is:

$$ x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \left[f\big(s,x(s)\big) + g\big(s,x(s)\big)\right] ds, $$

where $f, g \in C(J \times \mathbb{R}, \mathbb{R}),\ J = [t_0, T]$.

**Definition 2.2:** A function $\alpha = \alpha(t) \in C^q(J, \mathbb{R})$ is said to be a lower solution of Caputo FDE if

$ {}^cD^q\, \alpha(t) \le f(t,\alpha(t)),\ \alpha(t_0) \le x_0$ and

a function $\beta = \beta(t) \in C^q(J,\mathbb{R})$ an upper solution of Caputo FDE if ${}^cD^q\, \beta(t) \ge f(t,\beta(t)),\ \beta(t_0) \ge x_0$

**Definition 2.3:** Let $\alpha, \beta \in C^q(J, \mathbb{R})$ these are said to be:

(a) Natural lower and upper solution of (1) if ${}^cD^q\, \alpha \le f(t,\alpha) + g(t,\alpha),\ \alpha(t_0) \le x_0$
${}^cD^q\, \beta \ge f(t,\beta) + g(t,\beta),\ \beta(t_0) \ge x_0$

(b) Coupled lower and upper solution of type I for (1) if ${}^cD^q\, \alpha \le f(t,\alpha) + g(t,\beta),\ \alpha(t_0) \le x_0$
${}^cD^q\, \beta \ge f(t,\beta) + g(t,\alpha), \beta(t_0) \ge x_0$

(c) Coupled lower and upper solution of type II for (1) if ${}^cD^q\, \alpha \le f(t,\beta) + g(t,\alpha),\ \alpha(t_0) \le x_0$
${}^cD^q\, \beta \ge f(t,\alpha) + g(t,\beta),\ \beta(t_0) \ge x_0$

(d) Coupled lower and upper solution of type III for (1) if ${}^cD^q\, \alpha \le f(t,\beta) + g(t,\beta),\ \alpha(t_0) \le x_0$
${}^cD^q\, \beta \ge f(t,\alpha) + g(t,\alpha),\ \beta(t_0) \ge x_0$

**Lemma 2.4:** Let $p = p(t) \in C^q(J,\ \mathbb{R}),\ {}^cD^q\, p \le -\sigma(t)p$, where $\sigma(t) \le M, M > 0$ and $p(0) \le 0$, then $p(t) \le 0$ on $J$

See[5] for details of the proof.

**Theorem**: Let $\alpha, \beta \in C^q(J, \mathbb{R})$ be lower and upper solution of (1), respectively and $f(t,u)$ and $g(t,u)$ satisfies the following one-sided Lipschitz condition, $f(t,\alpha_1) - f(t,\alpha_2) \le K_1(\alpha_1 - \alpha_2)$ and $g(t,\alpha_1) - g(t,\alpha_2) \le K_2(\alpha_1 - \alpha_2)$ where $\alpha_1 \ge \alpha_2$ and $K_1, K_2 > 0$, are constants, then $\alpha(t) \le \beta(t)$ on $J$, provided that $\alpha(t_0) \le \beta(t_0)$.

See[5] for details of the proof.

## 3. Main results

In the following section we establish the relation related to lower and upper solutions and establish the existence solution for Caputo fractional differential equation of type III.

**3.1Theorem:** Let $\alpha_0, \beta_0 \in C^q(J, \mathbb{R})$ and
${}^cD^q\, \alpha_0 \le f(t,\beta_0) + g(t,\beta_0),\ \alpha_0(t_0) \le x_0$

$$ {}^cD^q\, \beta_0 \ge f(t,\alpha_0) + g(t,\alpha_0),\ \ \beta_0(t_0) \ge x_0 $$

$$ \alpha_0(t) \le \beta_0(t) \text{ on } J,\ \ \alpha_0(t_0) \le x_0 \le \beta_0(t_0). $$

Suppose $f_x(t,x),\ g_x(t,x)$ exists in $'x'$ for each $'t'$, and $f_x(t,x)$ is increasing, $g_x(t,x)$ is decreasing in $'x'$ for all $'t'$. Assume that $f(t,x) \le f(t,y) + f_x(t,x)(x-y)$ for $x \ge y$ and $|f_x(t,x) -$

$f_x(t,y)| \le L_1|x-y|$and $g(t,x) \le g(t,y) + g_x(t,y)(x-y)$ for $x \ge y$ and $|g_x(t,x) - g_x(t,y)| \le L_2|x-y|$, then there exists monotone sequences $\{\alpha_n(t)\}$ , $\{\beta_n(t)\}$ which converge uniformly to the unique solution of (1) on $J$ and the convergence is quadratic.

**Proof**: Consider the coupled linear fractional differential equations given by:

$$^{c}D^{q}\alpha_{k+1} = f(t,\beta_k) + f_x(t,\beta_k)(\beta_{k+1} - \beta_k) + g(t,\beta_k) + g_x(t,\alpha_k)(\beta_{k+1} - \beta_k),\ \beta_{k+1}(t_0) = x_0 \quad (2)$$

$$^{c}D^{q}\beta_{k+1} = f(t,\alpha_k) + f_x(t,\beta_k)(\alpha_{k+1} - \alpha_k) + g(t,\alpha_k) + g_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k),\ \alpha_{k+1}(t_0) = x_0 \quad (3)$$

Since the right hand side satisfies a Lipschitz condition, there exist unique solutions $\alpha_{k+1}(t)$ and $\beta_{k+1}(t)$, corresponding to (2) and (3) respectively.

Our aim is to prove that $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_{k+1} \le \beta_{k+1} \le \cdots \le \beta_1 \le \beta_0$ on $J$.

We will show $\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0$ on $J$.

Setting $p = \alpha_1 - \beta_1$ on $J$, then $^{c}D^{q}\, p = {}^{c}D^{q}\, \alpha_1 - {}^{c}D^{q}\, \beta_1$. Using the assumption of theorem 3.1, (2) and (3) we get:

$$^{c}D^{q}\, p = [f(t,\beta_0) + f_x(t,\beta_0)(\beta_1 - \beta_0) + g(t,\beta_0) + g_x(t,\alpha_0)(\beta_1 - \beta_0)] - [\, f(t,\alpha_0) + f_x(t,\beta_0)(\alpha_1 - \alpha_0) + g(t,\alpha_0) + g_x(t,\alpha_0)(\alpha_1 - \alpha_0)] \quad (4)$$

but, $f(t,\beta_0) - f(t,\alpha_0) \le f_x(t,\beta_0)(\beta_0 - \alpha_0)$ and $g(t,\beta_0) - g(t,\alpha_0) \le g_x(t,\alpha_0)(\beta_0 - \alpha_0)$

$$\begin{aligned}\therefore {}^{c}D^{q}\, p \le f_x(t,\beta_0)(\beta_0 - \alpha_0) &\\ + f_x(t,\beta_0)(\alpha_0 - \alpha_1) &\\ + f_x(t,\beta_0)(\beta_1 - \beta_0) &\\ + g_x(t,\alpha_0)(\beta_0 - \alpha_0) &\\ + g_x(t,\alpha_0)(\beta_1 - \beta_0) &\\ + g_x(t,\alpha_0)(\alpha_0 - \alpha_1) &\end{aligned}$$

$$\le -[f_x(t,\beta_0) + g_x(t,\alpha_0)](\alpha_1 - \beta_1) \quad (5)$$

This inequality holds by assumption made in the theorem 3.1,
We have, $^{c}D^{q}\, p \le -[f_x(t,\beta_0) + g_x(t,\alpha_0)]p$, $p(0) = 0$. Using the Lemma 2.4,
we conclude $\alpha_1(t) \le \beta_1(t)$ (6)
Let $p = \alpha_0 - \alpha_1$ on $J$, then $^{c}D^{q}\, p = {}^{c}D^{q}\, \alpha_0 - {}^{c}D^{q}\, \alpha_1 \le f(t,\beta_0) + g(t,\beta_0) - [\, f(t,\beta_0) + f_x(t,\beta_0)(\beta_1 - \beta_0) + g(t,\beta_0) + g_x(t,\alpha_0)(\beta_1 - \beta_0)]$ (7)

$$^{c}D^{q}\, p \le -[f_x(t,\beta_0) + g_x(t,\alpha_0)](\beta_0 - \beta_1)$$
$$\le -[f_x(t,\beta_0) + g_x(t,\alpha_0)](\alpha_0 - \alpha_1)$$
$$\le -[f_x(t,\beta_0) + g_x(t,\alpha_0)]p, p(t_0) \le 0 \quad (8)$$

Using the Lemma 2.4, we conclude $\alpha_0(t) \le \alpha_1(t)$ on $J$.
Similarly, we obtain $\beta_1(t) \le \beta_0(t)$,
Thus, we have established $\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0$ on $J$.
Assuming for $k > 1$, $\alpha_0 \le \alpha_{k-1} \le \alpha_k \le \beta_k \le \beta_{k-1} \le \beta_0$ on $J$, we shall show $\alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k$ on $J$.

In order to do so, we set $p = \alpha_{k+1} - \beta_{k+1}$ and $^{c}D^{q}\, p = {}^{c}D^{q}\, \alpha_{k+1} - {}^{c}D^{q}\, \beta_{k+1}$

$$\begin{aligned} = f(t,\beta_k) + f_x(t,\beta_k)(\beta_{k+1} - \beta_k) + g(t,\beta_k) &\\ + g_x(t,\alpha_k)(\beta_{k+1} - \beta_k) &\\ - [\, f(t,\alpha_k) &\\ + f_x(t,\beta_k)(\alpha_{k+1} - \alpha_k) + g(t,\alpha_k) &\\ + g_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k)] &\end{aligned}$$

but $f(t,\beta_k) - f(t,\alpha_k) \le f_x(t,\beta_k)(\beta_k - \alpha_k)$ and

$g(t,\beta_k) - g(t,\alpha_k) \le g_x(t,\alpha_k)(\beta_k - \alpha_k)$

$$\begin{aligned} ^{c}D^{q}\, p \le f_x(t,\beta_k)(\beta_k - \alpha_k) &\\ + f_x(t,\beta_k)(\alpha_k - \alpha_{k+1}) &\\ + f_x(t,\beta_k)(\beta_{k+1} - \beta_k) &\\ + g_x(t,\alpha_k)(\beta_k - \alpha_k) &\\ + g_x(t,\alpha_k)(\beta_{k+1} - \beta_k) &\\ + g_x(t,\alpha_k)(\alpha_k - \alpha_{k+1}) &\end{aligned}$$

$$\le -[f_x(t,\beta_k) + g_x(t,\alpha_k)](\alpha_{k+1} - \beta_{k+1})$$

$\le -[f_x(t,\beta_k) + g_x(t,\alpha_k)]p$, and $p(0) = 0$

Using the Lemma 2.4, we conclude $\alpha_{k+1}(t) \le \beta_{k+1}(t)$ on J.

Finally, we shall show that $\alpha_k(t) \le \alpha_{k+1}(t)$
Setting $p = \alpha_k - \alpha_{k+1}$ , then $^{c}D^{q}\, p = {}^{c}D^{q}\, \alpha_k - {}^{c}D^{q}\, \alpha_{k+1}$ using (2), we get

$$\begin{aligned} ^{c}D^{q}\, p = f(t,\beta_{k-1}) &\\ + f_x(t,\beta_{k-1})(\beta_k - \beta_{k-1}) &\\ + g(t,\beta_{k-1}) &\\ + g_x(t,\alpha_{k-1})(\beta_k - \beta_{k-1}) &\\ - [f(t,\beta_k) &\\ + f_x(t,\beta_k)(\beta_{k+1} - \beta_k) + g(t,\beta_k) &\\ + g_x(t,\alpha_k)(\beta_{k+1} - \beta_k)] &\end{aligned}$$

but $f(t,\beta_{k-1}) - f(t,\beta_k) \le f_x(t,\beta_{k-1})(\beta_{k-1} - \beta_k)$ and $g(t,\beta_{k-1}) - g(t,\beta_k) \le g_x(t,\beta_{k-1})(\beta_{k-1} - \beta_k)$, $\beta_{k-1} \ge \beta_k$

$$\begin{aligned}\therefore\ {}^{c}D^{q}\, p \le f_x(t,\beta_{k-1})(\beta_{k-1} - \beta_k) &\\ + f_x(t,\beta_{k-1})(\beta_k - \beta_{k-1}) &\\ + f_x(t,\beta_k)(\beta_k - \beta_{k+1}) &\\ + g_x(t,\beta_{k-1})(\beta_{k-1} - \beta_k) &\\ + g_x(t,\alpha_{k-1})(\beta_k - \beta_{k-1}) &\\ + g_x(t,\alpha_k)(\beta_k - \beta_{k+1}) &\end{aligned}$$

by hypotheses, $g_x(t,\alpha_k) \leq g_x(t,\beta_{k-1})$ and $g_x(t,\alpha_{k-1})(\beta_{k-1}-\beta_k) \leq g_x(t,\beta_{k-1})(\beta_{k-1}-\beta_k)$
Thus we get, ${}^cD^q\, p \leq [f_x(t,\beta_k) + g_x(t,\alpha_k)]\, (\beta_k - \beta_{k+1}) \leq -[f_x(t,\beta_k) + g_x(t,\alpha_k)](\alpha_k - \alpha_{k+1}) \leq -[f_x(t,\beta_k) + g_x(t,\alpha_k)]p,\ \ p(0) = 0$
Using the Lemma 2.4, we conclude $\alpha_k(t) \leq \alpha_{k+1}(t)$.
Similarly, we can establish $\beta_{k+1} \leq \beta_k$ on $J$. Hence by induction we have $\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k$ on $J$.
The sequences are uniformly bounded and using the Ascoli-Arezela theorem, we conclude that there is a subsequence that converges uniformly on J. This provides monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ that converge to the unique solution of Caputo fractional differential equation (1).
To establish the quadratic convergence of $\{\alpha_n(t)\}, \{\beta_n(t)\}$ to the unique solution,

Set $p_{n+1}(t) = x(t) - \alpha_n(t) = x - \alpha_n$ and $q_{n+1}(t) = \beta_n(t) - x(t) = \beta_n - x$, then consider
${}^cD^q p_{n+1} = {}^cD^q x - {}^cD^q \alpha_{n+1}$

$$\begin{aligned}{}^cD^q p_{n+1} = {} & f(t,x) + g(t,x) - f(t,\beta_n) \\ & - f_x(t,\beta_n)(\beta_{n+1} - \beta_k) - g(t,\beta_n) \\ & - g_x(t,\alpha_n)(\beta_{n+1} - \beta_n)\end{aligned}$$

By using mean value theorem, using the increasing nature of $f_x$ and the decreasing nature of $g_x$ we get

$$\begin{aligned}{}^cD^q p_{n+1} = {} & f_x(t,\xi)(x - \beta_n) + g_x(t,\eta)(x - \beta_n) \\ & - [f_x(t,\beta_n) + g_x(t,\alpha_n)](\beta_{n+1} \\ & - x + x - \beta_k)\end{aligned}$$

$= -f_x(t,\xi)q_n - g_x(t,\eta)q_n - [f_x(t,\beta_n) + g_x(t,\alpha_n)](\, q_{n+1} - q_n)$, where $x < \xi, \eta < \beta_n$

$= [f_x(t,\beta_n) - f_x(t,\xi)]q_n + [g_x(t,\alpha_n) - g_x(t,\eta)]q_n - [f_x(t,\beta_n) + g_x(t,\alpha_n)]\, q_{n+1}$

$\leq [f_x(t,\beta_n) - f_x(t,x)]q_n + [g_x(t,\alpha_n) - g_x(t,x)]q_n + K_1 q_{n+1} + K_2 q_{n+1}$

where $|f_x(t,\beta_n)| \leq K_1$ and $|g_x(t,\alpha_n)| \leq K_2$

$\leq L_1|\beta_n - x|q_n + L_2|\beta_n - x|q_n + Kq_{n+1}$, as $\alpha_n \leq \beta_n$ and $K = Max(K_1, K_2)$ $= L|q_n|_0^2 + Kq_{n+1}$, here $L = Max(L_1, L_2)$ and $|q_n|_0 = \max\{q_n(t)\}, t_0 \leq t \leq T$

Thus we have, ${}^cD^q p_{n+1} \leq L|q_n|_0^2 + Kq_{n+1}$

A similar computation gives
${}^cD^q q_{n+1} \leq M|p_n|_0^2 + Np_{n+1}$, $M$ and $N$are constants as in earlier case.
The last two inequalities can be represented in vector form as follows:

Let $r_n = \begin{bmatrix} p_n \\ q_n \end{bmatrix}$, $A = \begin{bmatrix} 0 & K \\ N & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix}$, using this yields,

$${}^cD^q r_{n+1} \leq A|r_n|_0^2 + Br_{n+1}$$

This gives $r_{n+1} \leq A|r_n|_0^2 \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(B(t-s)^q)ds \ \leq N_0|r_n|_0^2$, where
$N_0 = L_{\frac{1}{q}}(T - t_0)^q E_{q,q}(B(t-s)^q)$
Thus we have the estimate $r_{n+1} \leq N_0|r_n|_0^2$ which gives quadratic convergence.

## 4 Conclusions:

The above result gives us the upper and lower solutions for nonlinear Caputo fractional differential equations. Computing the solution of nonlinear Caputo FDE is a challenge since it does not enjoy the properties of the integer order derivatives. Using this, it is possible to construct monotone sequence with quadratic convergence to the solution of the original problem.

## Author Statements:

- **Ethical approval:** The conducted research is not related to either human or animal use.
- **Conflict of interest:** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper
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- **Use of AI Tools:** The author(s) declare that no generative AI or AI-assisted technologies were used in the writing process of this manuscript.

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