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Research Article

Symmetry Analysis and Exact Traveling Wave Solutions of a Time-Fractional Higher order nonlinear Partial Differential Equations

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Lie Classical method

Abstract:

In the present paper, the invariant solutions of higher order time-fractional nonlinear partial differential equations namely, the sixth-order generalized Sawada-Kotera equation and seventh-order Korteweg-de Vries (KdV) equation. With the aid of conformable derivatives, symmetries are obtained and thereby reductions. The exact traveling wave solutions are obtained with the application of $\left(\frac{F}{G}\right)-$ Expansion Method to time-fractional higher order nonlinear partial differential equations. Novel general traveling wave solutions with arbitrary parameters are effectively presented in trigonometric, hyperbolic, and rational function forms.

1. Introduction

1.1 Scope

Fractional calculus across decades inspired researchers who have endeavored to redefine problems in sciences. Fractional derivative was originated in 1695 in a conversation between Leibnitz and L'Hospital. Later on Riemann-Liouville gave the method of representing the fractional derivatives [1, 2]. In the definition of fractional derivatives, they applied the findings of the gamma function and Cauchy integral formula. Some more definitions of fractional derivatives have been formed by Riesz [3], Caputo, Riemann, etc. The conformal derivative representation of fractional derivative is given by Khalil et al. [4] which satisfies all the properties of derivatives including chain rule formula. A number of methods are being applied to evaluate the exact solution of nonlinear partial differential equation in the literature [5-16]. In mathematics, differential equations study has been contributing substantially in modeling problems in real world and their applications not only in physics but in virtually all scientific and engineering disciplines. The nonlinear partial differential equations may include fractional and non-fractional derivatives. Any nonlinear partial differential equations change into fractional nonlinear partial differential equations only by taking the order of given differential equation in fractional number in place of integer. These fractional nonlinear partial differential equations are utilized in many physical events and many other domains of science like fluid mechanics, aerodynamics, nonlinear optics, plasma physics, hydrodynamics, optical fibres, biology, chemistry etc.

There is no certain method or means to solve any kind of non-linear partial differential equations of the same order, and to the great amount of the developed efforts the fractional nonlinear partial differential equations have been addressed,

resulting in the new strategy that will explain the system in everything. The methods include Lie symmetry method, the $(\frac{G'}{G})$ −expansion method, fractional sub-equation method, exp-function method, first integral method etc.
The significant approach for determining exact solution of nonlinear partial differential equations is Lie symmetry analysis [17-23]. Within the scope of this paper, we concentrate on the subsequent two time fractional nonlinear partial differential equations of higher order viz., sixth order generalized time fractional Swada-Kotera equation is given by

$$u_t^{\alpha}+ a\, u^2u_{xx}+b\, u_xu_{xxx}+c\, u\, u_{xxxx}+d\, u_{xxxxxx}=0. \quad (1.1.1)$$

Where $u = u(x, t), a, b, c, d$ are arbitrary constants and u_t^{α} is the conformal fractional derivative of order α with respect to 't'.And the seventh order time fractional Korteweg-de-Vries(KdV) equation is given by

$$u_t^{\alpha}+ a\, u^3u_x +bu^3+cuu_xu_{xx}+du^2u_{xxx} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x}=0. \quad (1.1.2)$$

Where $u = u(x, t), a, b, c, d, e, f$ and g are non-zero arbitrary constants. Here $u^{\alpha}{}_t$ is the conformal fractional derivative of order α with respect to 't'.

1.2 Related Work

Some researchers have added to this by deriving the exact solutions of Sawada-Koteraequation. Gazizov et al. a c q u i r e s y mmetrical characteristics in 2009. Wang G.W. and Xu T.Z.[24] address the exact solutions and analyze invariants of Sharma-Tasso-Olver equation of fixed order and arbitrary order nonlinear time fractional in 2013 using Lie group approach. Lie symmetries of time fractional Caudrey-Dodd-Gibbon-Swada-Kotera equation has been deducted by Baleanua et al. [25]. Saberi et al.[26] have derived lie symmetry analysis and exact solutions of the time fractional generalized Hirota-Satusuma coupled KdV system. Zhang H. and Jinag X.Y. gave convergent numericalmethod for the two-dimensional nonlinear time fractional diffusion-wave equation. In 2019 Saleh et al. f r a me d t h e a n s w e r s t o n o n l i n e a r fractional partial differential equations in terms of the singular manifold method. Some similarity and numerical solutions of the time fractional Burgers System were acquired by Zhang et al.[27] obtained. Yuhang Wang and Lianzhong Li[28] in 2019 obtained Lie symmetry and exact solution of sixth-order generalized time-fractional swada-kotera equation. In 2020 Roul et al.[29] gave a high order numerical method and its convergence for time-fractional fourth order partial differential equations.
In 1877 Boussinesq firstly introduced KdV equation and then in 1895 it was rediscovered by Gustav and Diederik Kortweg. In 1965 Zabusky and Kruskal [30] studied the behavior of solutions of the KdVequatons. In 1967 they developed inverse scattering transform. In 2005 Geyikli and Kaya[31], in 2006 Helal, Yan and mehanna[32], in 2007 Li and Wang[33] gave the number of analytical solutions of the KdV equation determining successfully by using finite difference schemes, finite element scheme and fourier spectral method.

1.3 Motivation

The related work above discussed led to motivate us for solving the time fractional higher order nonlinear partial differential equations. These equations come into view of many distinctive physical environments. It has been used in conformable field theory. It is of interest to use to find symmetry, exact wave solution by the series method and $(\frac{F}{G})$ -expansion method.

1.4 Outline of the present work

Introduction has been presented in section I. The section II gives preliminaries of conformal derivatives and methodology of $(\frac{F}{G})$- Expansion Method, Section III present the Lie symmetries analysis and Formation of the solutions(Series solution, wave solution) of time fractional Swada-Kotera equation , the section IV contains Lie symmetries analysis and Formation of the solutions(Series solution, wave solution) of time fractional seventh order time fractional Korteweg-de-Vries(KdV) equation , and our conclusion are outlined in part V.

2. Preliminaries

In this section, the conformal derivative has been presented.

2.1 Conformable Derivative

Definition: If j is a real-value function in the domain [0, ∞), i.e., j : [0, ∞) → $\mathcal{R}$. Thenthe conformable fractional derivative [20] of j of order $\alpha \in (0, 1]$ in the half space $u > 0$ is defined as

$$D^{\propto}(j)(u)= \lim_{\varepsilon\to 0}\frac{j(u+\varepsilon u^{1-\alpha})-j(u)}{\varepsilon}. \quad (2.1.1)$$

Characteristics of the conformable fractional derivative given as:

Let φ, ψ be two α-differentiable at point u>0 and α∊ (0,1].Then

1. $D^{\alpha}(a\varphi + b\psi) = a(D^{\alpha}\varphi)(u) + b(D^{\alpha}\psi)(u)$, for all a,b∊ (0, 1].
2. $D^{\alpha}(u^p) = pu^{p-\alpha}$, for all p∊R.
3. $D^{\alpha}(k) = 0$, for any constant k.
4. $D^{\alpha}(\varphi\psi)(u) = \varphi(u)D^{\alpha}(\psi)(u) + \psi(u)D^{\alpha}(\varphi)(u)$.
5. $D^{\alpha}\left(\frac{\varphi}{\psi}\right)(u) = \frac{\varphi(u)D^{\alpha}(\psi)(u) - \psi(u)D^{\alpha}(\psi)(u)}{\psi^2(u)}$.
6. If φ is differentiable then $D^{\alpha}(\varphi)(u) = u^{1-\alpha}\frac{d}{dt}(\varphi(u))$.

2.2 Methodology of the $\left(\frac{F}{G}\right)$ −Expansion Method

Taking into account a nonlinear partial differential equation expressed as

$$\kappa(d, d_s, d_t, d_{ss}, d_{tt}, d_{ts}, \cdots) = 0, \qquad (2.2.1)$$

Where $d = d(t,s)$ is an unknown function and k is a polynomial in $d(t,s)$ and its partial derivatives, involving nonlinear terms and higher order derivatives. The key steps of $\left(\frac{F}{G}\right)$-Expansion Method are as under

Step 1. By defining the variable $\zeta = t - e\frac{s^{\alpha}}{\alpha}$, assume that

$$d(t,s) = d(\zeta), \zeta = t - e\frac{s^{\alpha}}{\alpha} \qquad (2.2.2)$$

Where e is nonzero constant, the wave variable ζ transform equation (2.2.1) to ODE

$$\kappa(d, d', d'', \cdots) = 0. \qquad (2.2.3)$$

Step 2. The ODE has a solution which (2.2.3) can be represented by a polynomial in $\left(\frac{F}{G}\right)$ as

$$b(\zeta) = \sum_{i=0}^{y} l_i \left(\frac{F}{G}\right)^i \qquad (2.2.4)$$

Where $G = G(\zeta)$ and $F = F(\zeta)$ satisfy the FLODS in the form

$$F'(\zeta) = \eta G(\zeta),\ G'(\zeta) = \rho F(\zeta). \qquad (2.2.5)$$

Here $l_0, l_1, \cdots, l_y$, η and ρ are constants that we will have to find out later and $l_y \neq 0$. We can find the positive integer, say p homogeneous balancing method to the highest order derivatives terms and nonlinear terms appearing in ordinary differential equation.

With the help of (2.2.5), the solutions can be computed $F(\zeta)$ and $G(\zeta)$ as follows:

Case-I: If $\eta > 0$ and $\rho > 0$, then (2.2.5) solutions can be represented by hyperbolic function which can be represented as:

$$\begin{cases} F(\zeta) = C_1\cosh(\sqrt{\eta}\sqrt{\rho}\zeta) + C_2\frac{\sqrt{\eta}}{\sqrt{\rho}}\sinh(\sqrt{\eta}\sqrt{\rho}\zeta), \\ G(\zeta) = C_1\frac{\sqrt{\rho}}{\sqrt{\eta}}\sinh(\sqrt{\eta}\sqrt{\rho}\zeta) + C_2\cosh(\sqrt{\eta}\sqrt{\rho}\zeta). \end{cases}$$

(2.2.6)

Case-II: If $\eta < 0$ and $\rho < 0$, then (2.2.5) also has the hyperbolic function solutions:

$$\begin{cases} F(\zeta) = C_1\cosh(\sqrt{-\eta}\sqrt{-\rho}\zeta) - C_2\frac{\sqrt{\eta}}{\sqrt{\rho}}\sinh(\sqrt{-\eta}\sqrt{-\rho}\zeta), \\ G(\zeta) = -C_1\frac{\sqrt{-\rho}}{\sqrt{-\eta}}\sinh(\sqrt{-\eta}\sqrt{-\rho}\zeta) + C_2\cosh(\sqrt{-\eta}\sqrt{-\rho}\zeta). \end{cases}$$

(2.2.7)

Case-III: If $\eta > 0$ and $\rho < 0$, then (2.2.5) has the trigonometric solutions:

$$\begin{cases} F(\zeta) = C_1\cos(\sqrt{\eta}\sqrt{-\rho}\zeta) + C_2\frac{\sqrt{\eta}}{\sqrt{-\rho}}\sin(\sqrt{\eta}\sqrt{-\rho}\zeta), \\ G(\zeta) = -C_1\frac{\sqrt{-\rho}}{\sqrt{\eta}}\sin(\sqrt{\eta}\sqrt{-\rho}\zeta) + C_2\cos(\sqrt{\eta}\sqrt{-\rho}\zeta). \end{cases}$$

(2.2.8)

Case-IV: If $\eta < 0$ and $\rho > 0$, then (2.2.5) again obtain solutions in the form of trigonometric function:

$$\begin{cases} F(\zeta) = C_1\cos(\sqrt{-\eta}\sqrt{\rho}\zeta) + C_2\frac{\sqrt{-\eta}}{\sqrt{\rho}}\sin(\sqrt{-\eta}\sqrt{\rho}\zeta), \\ G(\zeta) = -C_1\frac{\sqrt{\rho}}{\sqrt{-\eta}}\sin(\sqrt{-\eta}\sqrt{\rho}\zeta) + C_2\cos(\sqrt{-\eta}\sqrt{\rho}\zeta). \end{cases}$$

(2.2.9)

Step 3. Substituting (2.2.4) and (2.2.5) into equation (2.2.3), obtain a system of algebraic equations in $\left(\frac{F}{G}\right)$ $(i = 1, 2, 3, \cdots, y)$. Equating the coefficients of $\left(\frac{F}{G}\right)^i$ to zero yields a set of nonlinear algebraic equations in $a_j(j = 0, 1, 2, \cdots, y)$ and d. Solving these nonlinear algebraic system of equations by Maple, we obtain exact solutions of (2.2.1).

3. Lie Symmetry

We consider nonlinear partial differential equations in two variables in form given below:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = F(t, x, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx}, u_{xxxxxx}) = 0.$$

(3.1)

Here $\alpha \in (0, 1]$ is a parameter.
The infinitesimal of the one parameter lie group is as follows:

$$u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2),$$

$$x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2),$$

$$t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2),$$

Here ϵ is the group parameter. (3.2)

This transformation leaflets the set of solutions of equation invariant and gives a linear sys-tem of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$, $\eta(x,t,u)$.
From above transformations, we obtained vector field which can be written as

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \quad (3.3)$$

Where τ, ξ and η are the undermined functions. By applying lie transformation to equation (1.1.1) we obtain the system of determining equations:

(i) $\tau_x = 0, \tau_u = 0$

(ii) $\xi_u = 0, \eta_{uu} = 0$

(iii) $6d\eta_{xu} - 15d\xi_{xx} = 0$

(iv) $2cu\xi_x + \eta c - 20d\xi_{xxx} + 15d\eta_{xxu} = 0$

(v) $4cu\eta_{xu} + b\eta_x - 6cu\xi_{xx} - 15d\xi_{xxxx} = 0$, $b\eta_u + 2b\xi_x = 0$

(vi) $6cu\eta_{xxu} + 2\eta au + 4au^2\xi_x - 4cu\xi_{xxx} + 15d\eta_{xxxxu} - 6d\xi_{xxxx} = 0, -3b\xi_{xx} + 3b\eta_{xu} = 0$

(vii) $3b\eta_{xxu} - b\xi_{xxx} = 0$

(viii) $2au^2\eta_{xu} - cu\xi_{xxxx} + 6d\eta_{xxxxxu} + bt^{(1-\alpha)}\xi_t\eta_{xxx} - d\xi_{xxxxxx} - au^2\xi_{xx} = 0$

(ix) $-\tau\alpha t^{-\alpha} + \tau t^{-\alpha} + t^{(1-\alpha)}\eta_u - t^{(1-\alpha)}\eta_t = 0$

(x) $d\eta_{xxxxxx} + au^2\eta_{xx} + cu\eta_{xxxx} + t^{(1-\alpha)}\eta_t = 0$

(xi) $6\xi_x - \eta_u = 0$ (3.4)

On solving above equations we get the values of η, ξ, τ

$$\eta = uc_1$$

$$\xi = \frac{-c_1}{2}x + c_2$$

$$\tau = \frac{-3c_1 t}{\alpha} + c_3 t^{(1-\alpha)}, \quad (3.5)$$

Where c_1, c_2 and c_3 are arbitrary constants. Hence, the infinitesimal symmetry group for equation (1.1.1) is generated by the following three vector fields.

$$V_1 = u\frac{\partial}{\partial u} - \frac{x}{2}\frac{\partial}{\partial x} - \frac{3t}{\alpha}\frac{\partial}{\partial t}$$

$$V_2 = \frac{\partial}{\partial t}$$

$$V_3 = t^{(1-\alpha)}\frac{\partial}{\partial t}. \quad (3.6)$$

Using the above vector fields V_1, V_2, V_3 we obtain the similarity reductions of equation (1.1.1)

Case-I:

$$\frac{2dx}{-x} = \frac{\alpha dt}{-3t} = \frac{du}{u}$$

On solving, we get

$$\xi = xt^{\frac{-\alpha}{6}}, u = t^{\frac{-\alpha}{3}}Q(\xi)$$

Putting all values in equation (1.1.1), we get ordinary differential equation:

$$\frac{-\alpha}{3}Q(\xi) - \frac{\alpha}{6}\xi Q'(\xi) + aQ^2(\xi)Q''(\xi) + bQ'(\xi)Q'''(\xi) + cQ(\xi)Q''''(\xi) + dQ''''''(\xi) = 0. \quad (3.7)$$

Case-II:

$$\frac{dx}{1} = \frac{dt}{t^{(1-\alpha)}} = \frac{du}{0}$$

On solving, we get

$$x = \frac{t^\alpha}{\alpha}, u = Q(\xi). \text{ Here } \xi = x - v\frac{t^\alpha}{\alpha}. \quad (3.8)$$

Using these values in equation (1.1), we get ordinary differential equation:

$$dQ''''''(\xi) + cQ(\xi)Q''''(\xi) + bQ'(\xi)Q'''(\xi) + aQ^2(\xi)Q''(\xi) - vQ'(\xi) = 0. \quad (3.9)$$

3.2 Formation of the solutions

3.2.1 **Series Solution**

Theorem: Equation (3.7) admits the following power series solution of the form

$$Q(\xi) = B\xi^r \quad (3.2.1.1)$$

Where 'B' and 'r' are constants to be determined.
We equate the exponents of ξ suitably such that

their respective coefficients becomes zero. By equating the exponents 2r-3 and r-5, we get r=-2 and also, by equating the exponents r+1 and 2r-3, we obtain r=4.

Collecting the powers of ξ, we get

$$r = -2, r = 4 \qquad (3.2.1.2)$$

The value of B evaluated using maple software

$$B = \left(\frac{-210\, d}{2b+5c}\right) \qquad (3.2.1.3)$$

Using the value of B and ξ in equation (3.2.1.1), we get

$$Q(\xi) = \left(\frac{-210\, d}{2b+5c}\right) x t^{\frac{-\alpha}{6}} \qquad (3.2.1.4)$$

and represented by

$$u(\xi) = x t^{\frac{-\alpha}{6}} \left(\frac{-210\, d}{2b+5c}\right). \qquad (3.2.1.5)$$

3.2.2 Exact Traveling Wave Solutions

Here, we use equation (3.8) to reduce equation (1.1.1) to the nonlinear fractional ODE below

$$dQ''''''(\xi) + cQ(\xi)Q''''(\xi) + bQ'(\xi)Q'''(\xi) + aQ^2(\xi)Q''(\xi) - vQ'(\xi) = 0. \quad (3.2.2.1)$$

Balancing the highest order derivative term $Q''''''(\xi)$ with nonlinear term $Q(\xi)Q''''(\xi)$,

we get $m = 2$, and the resulting solution of (4.6) as

$$Q(\xi) = a_0 + a_1\left(\frac{F}{G}\right) + a_2\left(\frac{F}{G}\right)^2 \qquad (3.2.2.2)$$

Where a_0, a_1 and a_2 are constants to be determined. Using the equation (3.2.2.2) in the equation (3.2.2.1), collecting all terms with the same powers of $\left(\frac{F}{G}\right)$ together, and equating each coefficient of them to zero yields a set of algebraic equations. Solving these algebraic equations by Maple, we obtain the following results:

Case-I

When $a_2 = 0, a_1 = \frac{-v}{2b\mu\lambda^2}, a_0 = \frac{2(2c+\sqrt{4c^2-34ad})\mu\lambda}{a}$

(a) If $\lambda > 0$ and $\mu > 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = \frac{2(2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{C_1 cosh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2\frac{\sqrt{\lambda}}{\sqrt{\mu}} sinh(\sqrt{\lambda}\sqrt{\mu}\zeta)}{C_1\frac{\sqrt{\mu}}{\sqrt{\lambda}} sinh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2 cosh(\sqrt{\lambda}\sqrt{\mu}\zeta)}\right),$$
(3.2.2.3)

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

In specific, if $C_1 = 0$, $C_1 \neq 0$, we have the solitary wave solution.

$$Q(\xi) = \frac{2(2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{\sqrt{\lambda}}{\sqrt{\mu}} tanh(\sqrt{\lambda}\sqrt{\mu}\zeta)\right) \qquad (3.2.2.4)$$

(b)If $\lambda < 0$ and $\mu < 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = \frac{2(2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{C_1 cosh(\sqrt{-\lambda}\sqrt{\mu}\zeta) - C_2\frac{\sqrt{-\lambda}}{\sqrt{-\mu}} sinh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{-\lambda}} sinh(\sqrt{-\lambda}\sqrt{-\mu}\zeta) + C_2 cosh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)}\right),$$
(3.2.2.5)

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

In specific, if $C_1 = 0$, $C_1 \neq 0$, we have the solitary wave solution

$$Q(\xi) = \frac{2(2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{\sqrt{-\lambda}}{\sqrt{-\mu}} tanh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)\right) \qquad (3.2.2.6)$$

(C) If $\lambda > 0$ and $\mu < 0$, then the following trigonometric function solutions are obtained.

$$Q(\xi) = \frac{2(2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{C_1 cosh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2\frac{\sqrt{\lambda}}{\sqrt{-\mu}} sin(\sqrt{\lambda}\sqrt{-\mu}\zeta)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{\lambda}} sin(\sqrt{\lambda}\sqrt{-\mu}\zeta) + C_2 cos(\sqrt{\lambda}\sqrt{-\mu}\zeta)}\right),$$
(3.2.2.7)

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

(d) If $\lambda < 0$ and $\mu > 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = \frac{2(2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{C_1\cos(\sqrt{-\lambda}\sqrt{\mu}\zeta) - C_2\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\sin(\sqrt{-\lambda}\sqrt{\mu}\zeta)}{C_1\frac{\sqrt{\mu}}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}\sqrt{\mu}\zeta) + C_2\cos(\sqrt{-\lambda}\sqrt{\mu}\zeta)}\right), \quad (3.2.2.8)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

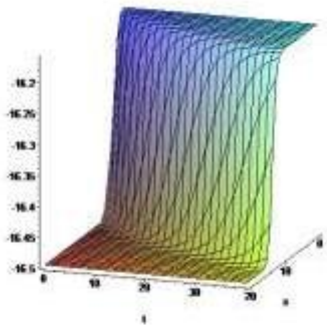

Figure 1. *Graph of solution given by Eq.* (3.2.2.4) *for x=0⋯* 20 *and t=0⋯* 40.

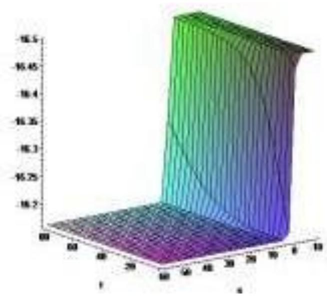

Figure 2. *Graph of Solutions given by Eq.* (3.2.2.6) *for x=-10⋯* 60 *and t=0⋯* 80.

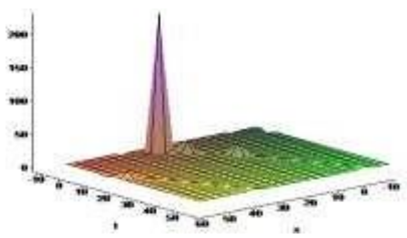

Figure 3. *Graph of Solutions given by Eq.* (3.2.2.7) *for x=−10⋯* 60 *and t=−10⋯* 60.

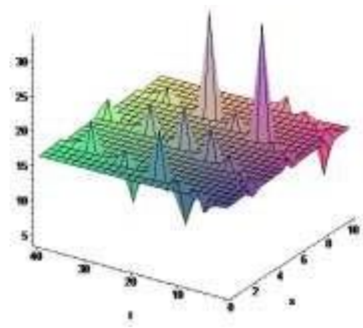

Figure 4. *Graph of Solutions given by Eq.* (3.2.2.8) *for x=0⋯* 10 *and t=0⋯* 40.

Case-II

When $a_2 = 0, a_1 = \frac{-v}{2b\mu\lambda^2}, a_0 = \frac{-2(-2c+\sqrt{4c^2-34ad})\mu\lambda}{a}$

(a) If $\lambda > 0$ and $\mu > 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = \frac{-2(-2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{C_1\cosh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2\frac{\sqrt{\lambda}}{\sqrt{\mu}}\sinh(\sqrt{\lambda}\sqrt{\mu}\zeta)}{C_1\frac{\sqrt{\mu}}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2\cosh(\sqrt{\lambda}\sqrt{\mu}\zeta)}\right), \quad (3.2.2.9)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

In specific, if $C_1 = 0$, $C_1 \neq 0$, we have the solitary wave solution.

$$Q(\xi) = \frac{-2(-2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{\sqrt{\lambda}}{\sqrt{\mu}}\tanh(\sqrt{\lambda}\sqrt{\mu}\zeta)\right) \quad (3.2.2.10)$$

(b) If $\lambda < 0$ and $\mu < 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = \frac{-2(-2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{C_1\cosh(\sqrt{-\lambda}\sqrt{-\mu}\zeta) - C_2\frac{\sqrt{-\lambda}}{\sqrt{-\mu}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{-\lambda}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\zeta) + C_2\cosh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)}\right), \quad (3.2.2.11)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

In specific, if $C_1 = 0$, $C_1 \neq 0$, we have the solitary wave solution

$$Q(\xi) = \frac{-2(-2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{\sqrt{-\lambda}}{\sqrt{-\mu}}\tanh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)\right) \quad (3.2.2.12)$$

(C) If $\lambda > 0$ and $\mu < 0$, then the following trigonometric function solutions are obtained.

$$Q(\xi) = \frac{-2(-2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{C_1\cosh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2\frac{\sqrt{\lambda}}{\sqrt{-\mu}}\sin(\sqrt{\lambda}\sqrt{-\mu}\zeta)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\sqrt{-\mu}\zeta) + C_2\cos(\sqrt{\lambda}\sqrt{-\mu}\zeta)}\right), \quad (3.2.2.13)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

(d) If $\lambda < 0$ and $\mu > 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = \frac{-2(-2c+\sqrt{4c^2-34ad})\mu\lambda}{a} + \frac{-v}{2b\mu\lambda^2}\left(\frac{C_1\cos(\sqrt{-\lambda}\sqrt{\mu}\zeta) - C_2\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\sin(\sqrt{-\lambda}\sqrt{\mu}\zeta)}{C_1\frac{\sqrt{\mu}}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}\sqrt{\mu}\zeta) + C_2\cos(\sqrt{-\lambda}\sqrt{\mu}\zeta)}\right) \quad (3.2.2.14)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

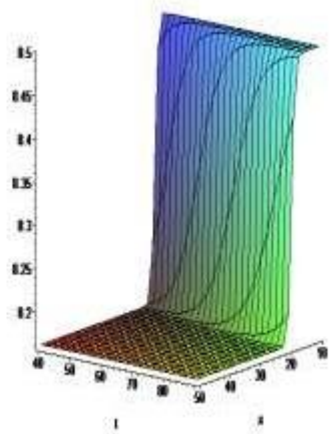

Figure5. *Graph of Solutions given by Eq.* (3.2.2.10) *for x=10⋯* 50 *and t=40⋯* 90.

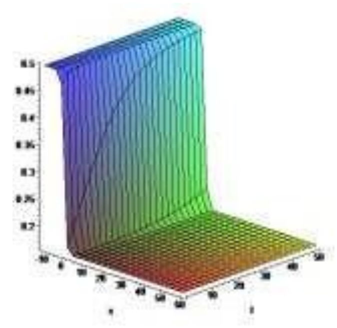

Figure 6. *Graph of Solutions given by Eq.* (3.2.2.12) *for x=*−10 ⋯ 60 *and t=0⋯* 50.

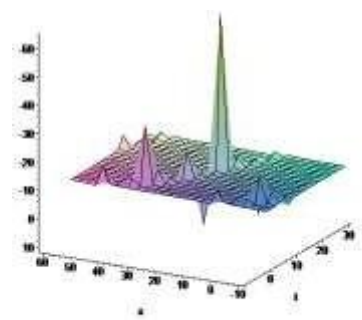

Figure 7. *Graph of Solutions given by Eq.* (3.2.2.13) *for x=*−10 ⋯ 60 *and t=*−10 ⋯ 30.

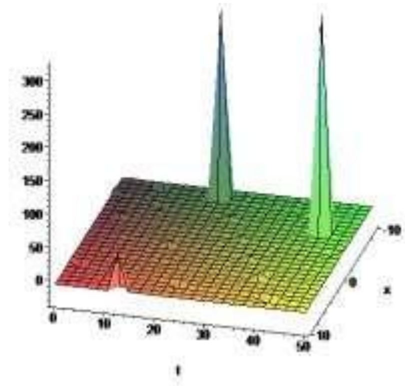

Figure 8. *Graph of Solutions given by Eq.* (3.2.2.14) *for x=*−10 ⋯ 10 *and t=*0 ⋯ 50.

4. Discussions

(i) Figure1 shows the solution given in Eq. (3.2.2.4) with parametric values

$$C_1 = 0, C_2 = 4, \lambda = 1, \mu = 1, \alpha = \frac{1}{2}, v = 1, a = -3, b = 3, c = 3, d = 3, x = 0 \cdots 20, t = 0 \cdots 40.$$

(ii) Figure2 shows the solution given in Eq. (3.2.2.6) with parametric values

$$C_1 = 0, C_2 = 1, \lambda = -1, \mu = -1, \alpha = \frac{1}{2}, v = 1, a = -3, b = 3, c = 3, d = 3, x = -10 \cdots 60, t = 0 \cdots 80.$$

(iii) ,Figure3 shows the solution given in Eq. (3.2.2.7) with parametric values

$$C_1 = 2, C_2 = 5, \lambda = -1, \mu = -1, \alpha = \frac{1}{2}, v = 1, a = -3, b = 4, c = 3, d = 3, x = -10 \cdots 60, t = -10 \cdots 60.$$

(iv) Figure4 shows the solution given in Eq. (3.2.2.8) with parametric values

$$C_1 = 2, C_2 = 5, \lambda = -1, \mu = 1, \alpha = \frac{1}{2}, v = 1, a = -3, b = 4, c = 3, d = 3, x = 0 \cdots 10, t = 0 \cdots 40.$$

(v) Figure5 shows the solution given in Eq. (3.2.2.10) with parametric values

$$C_1 = 0, C_2 = 4, \lambda = 1, \mu = 1, \alpha = \frac{1}{2}, v = 1, a = -3, b = 3, c = 3, d = 3, x = 10 \cdots 50, t = 40 \cdots 90.$$

(vi) Figure6 shows the solution given in Eq. (3.2.2.12) with parametric values

$$C_1 = 0, C_2 = 4, \lambda = -1, \mu = -1, \alpha = \frac{}{2}, v = 1, a = -3, b = 3, c = 3, d = 3, x = -10 \cdots 60, t = 0 \cdots 50.$$

(vii) Figure7 shows the solution given in Eq. (3.2.2.13) with parametric values

$$C_1 = 2, C_2 = 5, \lambda = 1, \mu = -1, \alpha = \frac{1}{2}, v = 1, a = -3, b = 3, c = 3, d = 3, x = -10 \cdots 60, t = -10 \cdots 30.$$

(viii) Figure4 shows the solution given in Eq. (3.2.2.14) with parametric values

$$C_1 = 2, C_2 = 5, \lambda = -1, \mu = 1, \alpha = \frac{1}{2}, v = 1, a = -3, b = 4, c = 3, d = 3, x = -10 \cdots 10, t = 0 \cdots 50.$$

5. Lie symmetries Analysis of Korteweg-de Vries (KdV) Equation

By applying lie transformation to equation (1.1.2) we obtain the system of determining equations:

(i) $\tau_x = 0, \tau_u = 0$

(ii) $\xi_u = 0, \eta_{uu} = 0, \xi_{uu} = 0, \xi_{xu} = 0$

(iii) $7\eta_{xu} - 21\xi_{xx} = 0$

(iv) $2gu\xi_x + g\eta - 35\xi_{xxx} + 21\eta_{xxu} = 0$

(v) $-10gu\xi_{xx} + f\eta_x + 5gu\eta_{xu} - 35\,\xi_{xxxx} = 0, f\eta_u + 2f\xi_x = 0$

(vi) $4du^2\xi_x + e\eta_{xx} - 10gu\xi_{xxx} + 2du\eta + 35\xi_{xxxu} - 21\xi_{xxxxx} + 10gu\eta_{xxu} = 0, 2e\xi_x + e\eta_u = 0, 4f\eta_{xu} + 2e\eta_{xu} - 6f\xi_{xx} - e\xi_{xx} = 0$

(vii) $-4f\xi_{xxx} + 3e\eta_{xxu} - e\xi_{xxx} + 6f\eta_{xxu} + 4cu\xi_x + c\eta + cu\eta_u = 0, -3du^2\xi_{xx} + cu\eta_x + e\eta_{xxx} - 5gu\xi_{xxxx} - 7\xi_{xxxxx} + 3du^2\eta_{xu} = 0, 3e\eta_{xu} - 3e\xi_{xx} = 0$

(viii) $3au^2\eta + cu\eta_{xx} + f\eta_{xxxx} - \xi_{xxxxxxx} - \frac{t\xi_t}{t^\alpha} + 6au^3\xi_x + 7\xi_{xxxxxxu} - du^2\xi_{xxx} + 3du^2\xi_x\eta_{xxu} - gu\xi_{xxxxx} + 5gu\eta_{xxxxu} = 0$

(ix) $4b\xi_x + 2b\eta_u = 0$

(x) $-f\xi_{xxxx} + 2cu\eta_{xu} + 3b\eta_x - cu\xi_{xx} = 0$

(xi) $t\eta_u + \tau - \tau\alpha - t\tau_t = 0$

(xii) $\frac{t\eta_t}{t^\alpha} + du^2\eta_{xxx} + gu\eta_{xxxxx} + \eta_{xxxxxxx} + au^3\eta_x = 0$

(xiii) $7\xi_x - \eta_u = 0.$ (4.1)

Whose solution is

$$\eta = uC_1$$

$$\xi = \frac{-C_1}{2}x + C_2$$

$$\tau = \frac{-7C_1 t}{2\alpha} + C_3 t^{1-\alpha} \quad (4.2)$$

Where C_1, C_2 and C_3 are arbitrary constants. Hence, the infinitesimal symmetry group for equation (1.2) is generated by the following three vector fields.

$$V_1 = u\frac{\partial}{\partial u} - \frac{x}{2}\frac{\partial}{\partial x} - \frac{7t}{2\alpha}\frac{\partial}{\partial t}$$

$$V_2 = \frac{\partial}{\partial x} \quad (4.3)$$

$$V_3 = t^{(1-\alpha)}\frac{\partial}{\partial t}.$$

Using the above vector fields V_1, V_2, V_3 we obtain the similarity reductions of equation (1.1.2) described in theorem 1 and 2.

Theorem 1: The transformation $\xi = xt^{\frac{-\alpha}{7}}$ and $u = t^{\frac{-2\alpha}{7}}Z(\xi), \xi = x - \frac{vt^\alpha}{\alpha}$ which is obtained from the similarity group method, reduce the equation (1.2) to ordinary differential equation as below;

$$\frac{-2\alpha}{7} Z(\xi) - \frac{\alpha}{7}\xi Z'(\xi) + aZ^3(\xi)Z'(\xi) + b(Z'(\xi))^3 + cZ(\xi)Z'(\xi)Z''(\xi) + dZ^2(\xi)Z'''(\xi) + eZ'(\xi)Z'''(\xi) + fZ'(\xi)Z''''(\xi) + gZ(\xi)Z'''''(\xi) + Z''''''(\xi) = 0.$$

having series solution. (4.4)

Proof

Equation (4.4) admits the following power series solution of the form

$$Q(\xi) = A\xi^p \quad (4.5)$$

Where A and p are constants to be determined. We equate the exponents of ξ suitably such that respective coefficients become zero. By equating the exponents $4p + 6$ and $2p + 2$, we get $p - 2$, by equating the exponents $2p + 2 = p + 7$, we obtain $p = 5$, and also by equating the exponents $3p + 4$ and $4p + 6$, we get $p = -2$.

Collecting the powers of ξ, we get

$$p = -2, p = 5 \quad (4.6)$$

The value of A evaluated using maple software

$$A = \sqrt{\left(\frac{10080}{2b+3c-6d}\right)} \quad (4.7)$$

Using the value of A and ξ in equation (4.5), we get

$$Q(\xi) = \sqrt{\left(\frac{10080}{2b+3c-6d}\right)}\, \left(xt^{\frac{-\alpha}{7}}\right)^{-2} \quad (4.8)$$

and represented by

$$u = \frac{1}{x^2} t^{\frac{-2\alpha}{7}} \sqrt{\left(\frac{10080}{2b+3c-6d}\right)} \quad (4.9)$$

Theorem 2:

The transformation $x = \frac{t^\alpha}{\alpha}, u = Z(\xi), \xi = x - \frac{vt^\alpha}{\alpha}$ which is obtained from the similarity group method, reduce the equation (1.1.2) to ordinary differential equation as below;

$$-vZ'(\xi) + aZ^3(\xi)Z'(\xi) + b(Z'(\xi))^3 + cZ(\xi)Z'(\xi)Z''(\xi) + dZ^2(\xi)Z'''(\xi) + eZ'(\xi)Z'''(\xi) + fZ'(\xi)Z''''(\xi) + gZ(\xi)Z'''''(\xi) + Z''''''(\xi) = 0. \quad (4.10)$$

Gives wave solution.

Proof: Balancing the highest order derivative term $Q''''''(\xi)$ with nonlinear term $Q(\xi)Q''''(\xi)$, we get $m = 2$, and the resulting solution of (4.7) as

$$Q(\xi) = a_0 + a_1\left(\frac{F}{G}\right) + a_2\left(\frac{F}{G}\right)^2 \quad (4.11)$$

Where a_0, a_1 and a_2 are constants to be determined. Using the equation (4.11) in the equation (4.10), collecting all terms with the same powers of $\left(\frac{F}{G}\right)$ together, and equating each coefficient of them to zero yields a set of algebraic equations. Solving these algebraic equations by Maple, we obtain the following results:

$$a_0 = a_0, a_1 = \pm\frac{\sqrt{-2\alpha(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}, a_2 = 0,$$

Where a_0 is an arbitrary constant.

Case-I When

$$a_0 = a_0,\ a_1 = \frac{\sqrt{-2\alpha(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a},\ a_2 = 0.$$

(a) If $\lambda > 0$ and $\mu > 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = a_0 + \left(\frac{\sqrt{-2\alpha(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{C_1 cosh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2\frac{\sqrt{\lambda}}{\sqrt{\mu}}sinh(\sqrt{\lambda}\sqrt{\mu}\zeta)}{C_1\frac{\sqrt{\mu}}{\sqrt{\lambda}}sinh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2 cosh(\sqrt{\lambda}\sqrt{\mu}\zeta)}\right) \quad (4.12)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

In specific, if $C_1 = 0$, $C_1 \neq 0$, we have the solitary wave solution.

$$Q(\xi) = a_0 + \left(\frac{\sqrt{-2\alpha(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right)\left(\frac{\sqrt{\lambda}}{\sqrt{\mu}}tanh(\sqrt{\lambda}\sqrt{\mu}\zeta)\right)$$

(4.13)

(b) If $\lambda < 0$ and $\mu < 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = a_0 + \left(\frac{\sqrt{-2\alpha(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{C_1 cosh(\sqrt{-\lambda}\sqrt{\mu}\zeta) - C_2\frac{\sqrt{-\lambda}}{\sqrt{-\mu}}sinh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{-\lambda}}sinh(\sqrt{-\lambda}\sqrt{-\mu}\zeta) + C_2 cosh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)}\right)$$

(4.14)

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

In specific, if $C_1 = 0$, $C_1 \neq 0$, we have the solitary wave solution

$$Q(\xi) = a_0 + \left(\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right)\left(\frac{\sqrt{-\lambda}}{\sqrt{-\mu}} tanh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)\right) \quad (4.15)$$

(c) If $\lambda > 0$ and $\mu < 0$, then the following trigonometric function solutions are obtained.

$$Q(\xi) = a_0 + \left(\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{C_1 cosh(\sqrt{\lambda}\sqrt{-\mu}\zeta) + C_2 \frac{\sqrt{\lambda}}{\sqrt{-\mu}} sin(\sqrt{\lambda}\sqrt{-\mu}\zeta)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{\lambda}} sin(\sqrt{\lambda}\sqrt{-\mu}\zeta) + C_2 cos(\sqrt{\lambda}\sqrt{-\mu}\zeta)}\right) \quad (4.16)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

(d) If $\lambda < 0$ and $\mu > 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = a_0 + \left(\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{C_1 cos(\sqrt{-\lambda}\sqrt{\mu}\zeta) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{\mu}} sin(\sqrt{-\lambda}\sqrt{\mu}\zeta)}{C_1 \frac{\sqrt{\mu}}{\sqrt{-\lambda}} sin(\sqrt{-\lambda}\sqrt{\mu}\zeta) + C_2 cos(\sqrt{-\lambda}\sqrt{\mu}\zeta)}\right) \quad (4.17)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

Case-II When

$$a_0 = a_0,\ a_1 = -\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a},\ a_2 = 0.$$

(a) If $\lambda > 0$ and $\mu > 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = a_0 + \left(-\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{C_1 cosh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2 \frac{\sqrt{\lambda}}{\sqrt{\mu}} sinh(\sqrt{\lambda}\sqrt{\mu}\zeta)}{C_1 \frac{\sqrt{\mu}}{\sqrt{\lambda}} sinh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2 cosh(\sqrt{\lambda}\sqrt{\mu}\zeta)}\right) \quad (4.18)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

In specific, if $C_1 = 0$, $C_1 \neq 0$, we have the solitary wave solution.

$$Q(\xi) = a_0 + \left(-\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right)\left(\frac{\sqrt{\lambda}}{\sqrt{\mu}} tanh(\sqrt{\lambda}\sqrt{\mu}\zeta)\right) \quad (4.19)$$

(b) If $\lambda < 0$ and $\mu < 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = a_0 + \left(-\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{C_1 cosh(\sqrt{-\lambda}\sqrt{\mu}\zeta) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{-\mu}} sinh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{-\lambda}} sinh(\sqrt{-\lambda}\sqrt{-\mu}\zeta) + C_2 cosh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)}\right), \quad (4.20)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

In specific, if $C_1 = 0$, $C_1 \neq 0$, we have the solitary wave solution

$$Q(\xi) = a_0 + \left(-\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{\sqrt{-\lambda}}{\sqrt{-\mu}} tanh(\sqrt{-\lambda}\sqrt{-\mu}\zeta)\right) \quad (4.21)$$

(c) If $\lambda > 0$ and $\mu < 0$, then the following trigonometric function solutions are obtained.

$$Q(\xi) = a_0 + \left(-\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{C_1\ cosh(\sqrt{\lambda}\sqrt{\mu}\zeta) + C_2 \frac{\sqrt{\lambda}}{\sqrt{-\mu}} sin(\sqrt{\lambda}\sqrt{-\mu}\zeta)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{\lambda}} sin(\sqrt{\lambda}\sqrt{-\mu}\zeta) + C_2 cos(\sqrt{\lambda}\sqrt{-\mu}\zeta)}\right) \quad (4.22)$$

Where $\zeta = x - v\frac{t^\alpha}{\alpha}$, C_1 and C_2 are arbitrary constants.

(d) If $\lambda < 0$ and $\mu > 0$, then the following hyperbolic solutions are obtained.

$$Q(\xi) = a_0 + \left(-\frac{\sqrt{-2a(-120\mu\lambda g - 14\mu\lambda e + 6da_0 + ca_0)\mu}}{a}\right) \times \left(\frac{C_1 \sqrt{\mu} \cos(\sqrt{-\lambda}\sqrt{\mu}\zeta) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{\mu}} \sin(\sqrt{-\lambda}\sqrt{\mu}\zeta)}{C_1 \frac{\sqrt{\mu}}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda}\sqrt{\mu}\zeta) + C_2 \cos(\sqrt{-\lambda}\sqrt{\mu}\zeta)}\right) \quad (4.23)$$

Where

$$\zeta = x - v\frac{t^\alpha}{\alpha},$$

C_1 and C_2 are arbitrary constants.

6. Conclusion

We obtain Lie symmetries, vector fields and symmetry reductions of sixth-order generalizedtime fractional Swada-Kotera equation and seventh order time fractional Korteweg-de-Vries (KdV) equation. Power series method and $(\frac{F}{G})$-expansion method yields exact travelling wave solutions of the equation. The identified solutions, which include hyperbolic, trigonometric, and rational categories, are potentially significant for various scientific fields. The present work highlights the Lie symmetry analysis and $(\frac{F}{G})$-expansion methods are significant mathematical tools to explore the nonlinear equations appearing in the field of physics and engineering.

Author Statements:

- **Ethical approval:** The conducted research is not related to either human or animal use.
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References

[1] Generalised KV.: Fractional calculus and applications. *Pitman research notes in math- ematics*, vol.301. London: Longman;(1994).

[2] IPodlubny I.(1999): Fractional differential equations. *San Diego: Academic Press.*

[3] Celik C. and Duman M., (2012). Crank-Nicolson method for the fractional equation with the Reisz fractional derivative. *J Comput Phys* 231:1743-1750.

[4] Khalil R., Horani M., Yousef A. and Sababhen M., (2014). A new definition of fractional derivative. *J Comput Appl Math* 264:65-70.

[5] Baleanu D., Inc M., Yusuf A. and Aliyu AI. (2017). Time fractional third-order evolution equation: symmetry analysis, explicit solutions, and conservation laws. *J Computation Nonlinear Dynamics* 13(2):021011.

[6] hang S. and Zhang HQ., (2011). Fractional sub-equation method and dummyTXdummy- its applications to nonlinear fractional PDEs. *Phys Lett A* 375(7):1069-73.

[7] Buckwar E. and Luchko Y., (1998). Invariance of a partial differential equation of fractional or- der under the lie group of scaling transformations. *J Math Appl Math* 227:81-97.

[8] Gazizov RK. and Kasatkin AA. and Lukashcuk SY., (2007). Continuous transformation groups of fractional differential equations. *Vestnik, USATU* 9:125-35.

[9] Griffiths DF., Dold JW. and Silvester DJ., (2010). Essential partial differential equations. *Springer*.

[10] Baleanu D., Inc M., Yusuf A. and Aliyu AI., (2017) Lie symmetry analysis, exact solutions and conservation laws for the time fractional modified Zakharov-Kuznetsov equation. *Nonlinear Anal Model Control* 22(6):861-76.

[11] Zheng B., (2012). G'/G-Expansion method for solving fractional partial differential equationsin the theory of mathematical physics. *Commun Theor Phys* 58:623-30.

[12] Gazizov RK., Kasatkin AA. and Lukashcuk SY., (2009). Symmetry properties of fractional diffusion equations. *Phys Scr* 136:014-16.

[13] Sahadevan R. and Bakkyaraj T., (2012). Invariant analysis of time fractional generalized burg- ers and Korteweg-de vries equations. *J Math Anal Appl* 393:341-7.

[14] Huang Q. and Zhdanov R., (2014). Symmetries and exact solutions of the time fractional Harry-Dym equation with Riemann-Liouville derivative. *Physica A* 409:110-18.

[15] Wang GW. and Xu TZ., (2014). Invariant analysis and exact solutions of nonlinear time fractional Sharma-Tasso-Olver equation by lie group analysis. Nonlinear Dyn 76:571- 80.

[16] Jefferson GF. and Carminati J., (2014). Fracsym automated symbolic computation of lie sym- metries of fractional differential equations. *Comput Phys*

Commun 185:430-41.

[17] Inc M., Yusuf A., Aliyu AI. and Baleanu D., (2018). Time-fractional Cahn-Allen and time- fractional Klein-Gordon equations: Lie symmetry analysis, explicit solutions and con- vergence analysis. *Physica A* 493:94-106.

[18] Bluman GW. and Kumei S., (1989). Symmetries and differential equations. *New York: Springer Verlag*.

[19] Olver PJ., (1993). Application of lie groups to differential equations. *New York: Springer-Verlag*.

[20] Ibragimov NH., (1999). Elementary lie group analysis and ordinary differential equations. Chichester: *John Wiley and Sons*.

[21] Adem AR. and Khalique CM., (2012). Symmetry reductions, exact solutions and conservation laws of a new coupled KdV system. *Commun Nonlinear Sci Numer Simul* 17:3465- 75.

[22] Gandarias ML.and Khalique CM., (2016). Symmetries, solutions and conservation laws of a class of nonlinear dispersive wave equations. *Commun Nonlinear Sci Numer Simul* 32:114-21.

[23] Kumar R., Kumar R., Bansal A., Biswas A., Yildirim A., Seithuti Moshokoa P. and Sirikr A A., (2023). Optical solitons and group invariants for Chen-Lee-Liu equation with time dependent chromatic dispersion and nonlinearity by Lie symmetry. *Journal of Physics and Optics* 4,24.

[24] Wang G. W. & Xu T. Z., (2013). Invariant analysis and exact solutions of nonlinear time-fractional Sharma–Tasso–Olver equation by Lie group analysis. *Nonlinear Dynamics,* 76(1), 571–580**.**

[25] Baleanu, D., Inc, M., Yusuf, A. & Aliyu, A. I., (2018). Lie symmetry analysis, exact solutions and conservation laws for the time fractional Caudrey–Dodd–Gibbon–Sawada–Kotera equation. *Communications in Nonlinear Science and Numerical Simulation*, 59, 222–234.

[26] Saberi, E. & Hejazi, S. R., (2018). Lie symmetry analysis, conservation laws and exact solutions of the time-fractional generalized Hirota–Satsuma coupled KdV system. *Physica A: Statistical Mechanics and Its Applications,* 492**,** 296–307.

[27] Zhang X. and Zhang Y., (2019). Some Similarity Solutions and Numerical Solutions to the Time-Fractional Burgers system *Symmetry***,** 11(1), 112.

[28] Yuhang Wang and Lianzhong Li., (2019). Lie Symmetry Analysis, Analytical Solution, and Conservation Laws of a Sixth-Order Generalized Time-Fractional Sawada–Kotera Equation**,** *Symmetry,* 11(12), 1436.

[29] Roul P. and Goura V. M. K. P., (2020). A high order numerical method and its convergence for time-fractional fourth order partial differential equations, *Applied Mathematics and Computation***,** Volume 366, Article 124727.

[30] Zabusky N. J. and Kruskal M. D., (1965). Interaction of Solitons in a Collisionless Plasma and the Recurrence of Initial States. *Physical Review Letters*, 15(6), 240–243.

[31] Turabi G. and Dogan K., (2005). An application for a Modified KdV equation by the decomposition method and finite element method, *Applied Mathematics and Computation*, 169(2), 971–981.

[32] Helal M. A. and Mehanna M. S., (2007). A Comparative Study between Two Different Methods for Solving the General Korteweg–de Vries Equation (GKdV), *Chaos, Solitons & Fractals,* 33(3),725–739.

[33] *Wang and Li.*, (2008). An Explicit Scheme for the KdV Equation, proposing a more stable finite-difference method. *Chinese Physics Letters,* 25(7).