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Research Article

Stability and Chaos Control in a Novel Three-Dimensional Multistable dynamical System with Coexisting Attractors

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Abstract:

This paper introduces three-dimensional Continuous-time autonomous dynamical system. We Construct new Lyapunove function for this system, the analysis of stability by new method is consistence with other method of stability. Basic dynamical proper ties such as equilibrium points, dissipativity, multistability, Wave form in time domain, phase portrait, bifurcation and Lyapunov exponents are studied, the analysis indicate that the system is unstable and hyperchaotic with Kaplan york dimension D_ky=2.1621. A novel feature of the system has multistability and attraction coexistence for two and three distinct initial condition sets. Also, adaptive control and synchronization system has been created, it is found that the hyperchaotic system achieved good results.

1. Introduction

Recently there has been increasing interest in nonlinear dynamical systems [1]. As a rule, Complexity occurs in dynamical system namely, System Internal microscopic or external microscopic motion affected by one or more forces, dynamical system may be conservative (Hamiltonian), they experience no energy loss, conversely system can be dissipative, which is the case in most real-life Situations, that involve losses [2]. Stability has become of great importance and focus of study for many researchers in recet times due to industrial and technological advance [3,4].
Rather than Chance Chaos is the ability to predict results, under Standing chaotic behavior has permeated every area of study in the modern era [5]. Numerous natural and scientific events exhibit chaotic motion, a common type of chaotic behavior, Chaotic dynamic according to many scientists is a fundemental Component in the understanding of these phenomena, numerous fields, including fluid mechanics, environmental science metrology, optic, heart and brian dynamics, epidemiology and illness research have detected chaotic motions [6,7].
Hyper chaos concept was firstly introduced in the seminal paper of Rössler to assert the dynamical patters of dynamical system when more than one positive Lyapunov exponent is found [8,9]. The discovery of many 3-D dynamical system such that Rabinovich system [10, 11, 12], sprott system [13], zhou system, etc [14,15].
One type of chaos treatment is chaos control, which falls into two categories: suppressing chaotic behaviour when it is harmful or an attempt to eradicate it, and creating and enhancing disorder when it is desired. Controlling a chaotic system is achieved through synchronisation [16,17,18,19]. Chaos synchronisation and control are crucial for studying nonlinear dynamical systems and are highly relevant for using chaos [20,21,22,23]

2. System Description

Recently, Safieddine Bouali constructed the new 3-D system, [24]. The system is described by

$$\begin{aligned}\dot{x} &= x(a-y)+\propto z\\ \dot{y} &= -y(b-x^2)\\ \dot{z} &= -x(c-\sigma z)-\beta z \quad \ldots\ldots\ldots\ldots (1)\end{aligned}$$

The variables x, y, and z typically represent states of the system.

Where $a = 4$, $b = \sigma = 1, c = 1.5$, $\propto = 0.3$, $\beta = 0,05$, and the initial condition (IC) of (x_0, y_0, z_0) = (0.5,0.5,0.5).

3. Properties of System (1)

3.1 System dissipativity

System (1) can be expressed in vector notations as

$$f = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{bmatrix}$$

Where the divergence of system (1) can be calculated using equation (2)

$$\nabla.\mathrm{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \dots\dots\dots\dots\dots\dots\dots..(2)$$

Where $f_1 = \dot{x},\ f_2 = \dot{y},\ f_1 = \dot{z}$

Take the parameter values as in system (1), we get:

$$\nabla.\mathrm{f} = x^2 + x - y + 2.95 \dots\dots\dots\dots\dots\dots..(3)$$

So, the dissipativity of system (1) is expressed in (3), as a variable rather than a Constant, it implies that system's (1) energy dissipation is not fixed, but depends on the system's (1) current condition. As a result, the system's nature is conservative for various beginning values and dissipative for the same ones.

3.2 Equilibrium points

Solving the following system of equations yielded the equilibrium points of system (1).

$f_1 = f_2 = f_3 = 0$, in (2) with $a = 4, b = \sigma = 1, c = 1.5, \alpha = 0.3, \beta = 0.05$. A calculation yields five equilibrium points one trivial equilibrium point $E_1 = (0,0,0) and$

$E_2 = (1, 0, -13.333)$, $E_3 = (-1, 0, 13,333), E_4 = (1, 0, 1.57), E_5 = (-1, 0, 1.42)$

3.3 Stability analysis

3.3.1 Characteristic equation

Linearizing system (1) a round Equilibrium point E with the aim to determine the Jacobian matrix J, it's corresponding eigen value λi, i=1,2,3,4 are found by solving the characteristic equation

$|J - \lambda I| = 0$ where I, the unit matrix.

The Jacobian matrix of new system (1) at E is given by

$$J(E) = \begin{bmatrix} a & -xy & \alpha \\ -2xy & (b - x^2) & 0 \\ -cx & 0 & \sigma x - \beta \end{bmatrix}$$

Thus, the Jacobian matrix of system (1) at E_1, is obtained as:

$$J(E_0) = \begin{bmatrix} 4 & 0 & 0.3 \\ 0 & -1 & 0 \\ -1.5 & 0 & -0.05 \end{bmatrix}$$

Using Matlab 2024 the characteristic equation of system (1) at E_1 is:

$$\lambda^3 - 2.9\lambda^2 - 3.7\lambda + 0.25 = 0 \dots\dots\dots\dots\dots\dots\dots..(4)$$

These are the eigenvalues:

$\lambda_1 = -1,\ \lambda_2 = 3.885, \lambda_3 = 0.064$

Thus, the trivial equilibrium E_1, of 3-D system (1) is hyperbolic a saddle-node, which is unstable.

Following the same methodology, it was found that the remaining equilibrium points E_2, E_3, E_4 and E_5 are unstable, and the result reported in Table (1). Hence the new system (1), is unstable.

3.3.2 Continued Fraction criteria,

The criterion was applied to system (1)'s characteristic equation (4) by creating a continuing fraction from equation (4)'s odd and even components.

$Q_1(\lambda) = \lambda^3 - 3.7\,\lambda$

$Q_2(\lambda) = -2.95\lambda^2 + 2.5$

To asses the fraction Q_1/Q_2, divide the denominator by the numerator, and then invert the reminder to get a continued fraction the way it is.

$$\frac{Q_1}{Q_2} = K_1\lambda + \cfrac{1}{K_2\lambda + \cfrac{1}{K_3\lambda + \cfrac{1}{K_4\lambda}}}$$

if all K_1, K_2, K_3 are positive, the roots of equation (4) will have negative real parts. since:

K_1 = - 0.339 < 0 K_2 = 0.816 > 0 K_3=-14.461

Therefore, system (1) is unstable. For the rest of equilibrium points $E_1, E_2, E_3, E_4\, and\, E_5$ we get System (1) unstable, and the results are given in Table (2).

Table 1. *Stability and classification detected Equilibrium points for new system (1)*

Equilibrium points	**The corresponding characteristics and eigenvalues**	**Type of Equilibria and stability**
E_1= (0,0,0,0)	λ^3-2.9 λ^2 -3.7 λ+0.25=0 (λ_1, λ_2, λ_3) = (-1,3.885,0.064)	Hyperbolic equilibrium point Unstable Node
E_2= (1,0, - 13.33)	λ^3-4.95 λ^2+8.24 λ=0 (λ_1, λ_2, λ_3) = (0,2.475 -1457i,2.475+1.457i)	Hyperbolic equilibrium point Unstable Focus
E_3= (- 1, 0, 13.33)	λ^3-2.95 λ^2-7.749 λ =0 (λ_1, λ_2, λ_3) = (-1.675,0.4625)	Non-Hyperbolic equilibrium Unstable Node
E_4= (1, 0, 1.57)	λ^3-4.95λ^2+3.779 λ =0 (λ_1, λ_2, λ_3) = (0,0.943,4.006)	Non-Hyperbolic equilibrium Unstable Point
E_5= (- 1, 0, 1.42)	λ^3-2.95 λ^2-4.176 λ =0 (λ_1, λ_2, λ_3) = (-1.045,0,3.995)	Non- Hyperbolic equilibrium Unstable Point

3.3.3 Lyapunov Function

We can use quadratic function of system (1)

$$V(x,y,z) = \frac{1}{2} * (x^2 + y^2 + z^2)$$
$$\dot{V}(x,y,z) = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial t} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Substitute system (1) in equation (5), we find

$$\dot{V}(x,y,z,) = 4x^2 - x^2y + 0.3xz + x^2y^2 - y^2 + xz^2 - 1.5xz - 0.05z^2 \ldots\ldots\ldots\ldots (6)$$

Substitute the initial condition in (6), we get $\dot{V}(x,y,z,) > 0$, also at the equilibrium points E_1, E_2, E_3, E_4 and E_5, the results are shown in Table (1). Hence system (1) is globally unstable.

3.3.4 A New Method for Lyapunov Function Construction Via Continued Fractions:

Step 1: Apply continued fraction criterion to create the factors of the characteristic polynomial.

$$Q(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots .+a_{n-1}\lambda + a_n = 0 \ldots\ldots\ldots\ldots\ldots (7)$$

Step 2: construct the Lyapunov Function as:

$$V(x) = \frac{1}{2}\sum_{j=1}^{n} |k_j| x_j^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

From the paragraph (3.3.2) we have

$$K_1 = 0.339, K_2 = 0.816 \; and \; K_3 = -14.461$$

With $x_1 = x, x_2 = y$ and $x_3 = z$ so the Lyapunov Function

$$V(x,y,z) = \frac{1}{2} (|k_1|x^2 + |k_2|y^2 + |k_3|z^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Step 3: The Lyapunov function, V must achieve the following condition for Stability of system

$$V(x,y,z) = 0 \Leftrightarrow (x,y,z) = (0,0,0)$$
$$V(x,y,z) > 0 \Leftrightarrow (x,y,z) \neq (0,0,0)$$
$$\dot{V}(x,y,z) < 0 \Leftrightarrow V(x,y,z) \neq (0,0,0)$$

The new criterion applied to all equilibrium points of system, we get $\dot{V}(x,y,z) > 0$, therefore the system (1) in not asymptotically stable and the results are given in table (2).

Table 2. *A summary of the stability of system (1) based on all the formentioned stability criteria.*

Equilibrium points	Lyapunov function $\dot{V}$	Continued Fraction Stability	New method for costructing Lyapunov Function via continued fractor

$E_1 = (0,0,0)$	0	$k_1 = -0.339$ $k_2 = -0.815$ $k_3 = -14.461$	0
$E_2 = (1,0,-13.333)$	148.0 (Unstable)	$k_1 = -0.2020$ $k_2 = -0.6007$ $unstable$	$2.0202x10^{-5}$ (Unstable)
$E_3 = (-1,0,-13.333)$	188.8 (Unstable)	$k_1 = -0.339$ $k_2 = 0.380694$ $unstable$	$3.38983x10^{-5}$ (Unstable)
$E_4 = (1,0,1.57)$	4.457 (Unstable)	$k_1 = -0.2020$ $k_2 = -1.3098$ $unstable$	0.903232 (Unstable)
$E_5 = (-1,0,1.42)$	7.62 (Unstable)	$k_1 = -0.339$ $k_2 = 0.706417$ $unstable$	1.221525 (Unstable)

For all the equilibrium points according to roots of characteristic equation, Lyapunov Function stability criteria, continued fraction and the new method for constructing Lyapunov function using continued fractions System (1) is not stable and this implies chaos.

3.4 Lyapunov Exponents

An essential tool for characterizing an attractor of a finite - dimensional nonlinear dynamical system and determining how sensitive it is beginning conditions are the Lyapunov exponents, when an n-dimensional hyper chaotic system has a higher number of Lyapunov exponents (n-2 positive Lyapunov exponents), it is more complex. Lyapunov exponent is a method for detecting chaos. [2]. For the starting state (0.5.05.05) and parameters (a, b, c, σ, α, β) = (4,1,1.5,1, 0.3,0.05), we compute the Lyapunov exponents of system (1) using MATLAB 24. The obtained Lyapunov exponents are

L_1=0.003903, L_2=0.098913 L_3=-0.634239 Because L_1, and L_2 are positive and L_3 is negative, therefore system (1) is hyperchaotic.

The Kaplan-York dimension D_{ky} can expressed as [25, 26]

$$D_{ky} = j + \frac{1}{L_{j+1}} \sum_{i=1}^{3} L_i < 0$$

meet both $\sum_{i=1}^{j} L_i > 0 \; and \; \sum_{i=1}^{3} L_i < 0$

For system (1) $\sum_{i=1}^{2} L_i$ = 0.102816>0 and $\sum_{i=1}^{3} L_i$=-0.5314234<0 with D_{ky} = 2.1621.

Figure (1) shows the dynamic of the Lyapunov foundations of the hyperchaotic system (1)

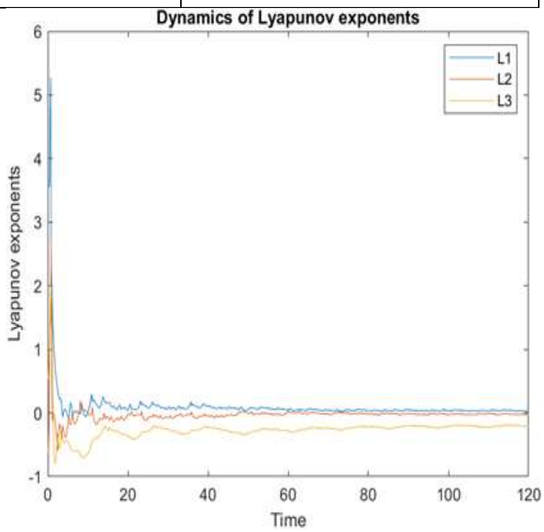


Figure 1. *Lyapunov Exponents of hyper chaotic system (1)*

3.5. Numerical simulation

For numerical simulation, we solved the 3D system (1) parameters value, and IC as in (1) using conventional fourth order Rung-kutta technique in MATLAB.

3.5.1 Waveform analysis

One of the fundamental features of chaotic dynamical systems is the non-periodic structure of the wave form of hyperchaotic system (1), as seen in Figures (2-4), shows the aperiodic waveforms of x_t, yt and z_t in time domain.

3.5.2 Phase Portraits Analysis

In this paragraph shows phase portraits of attractors of system (1) in (x versus y), (x versus z) and (y versus z) plane and in (x, y, z) space for (x, y, z) = (0.5,0.5,0.5).

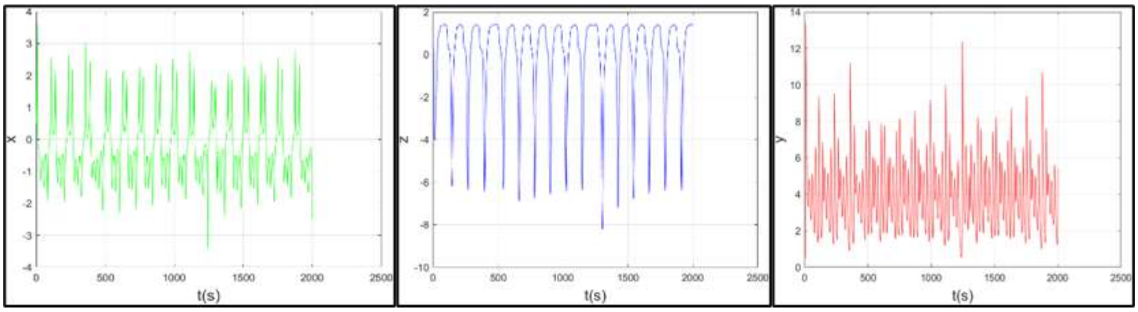


Figure 2. *Times versus x, Figure 3. Times Versus y, Figure 4. Times versus Z,*

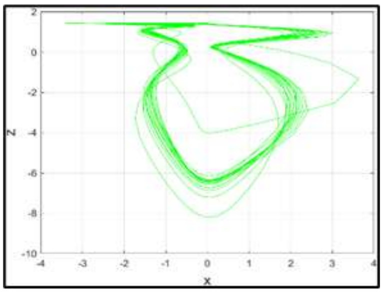


Figure 5. *2D phase plot of the attractors in (x -y) plane*

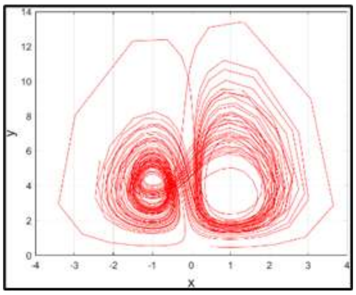


Figure 6. *2D phase plot of the attractors in (y -z) plane*

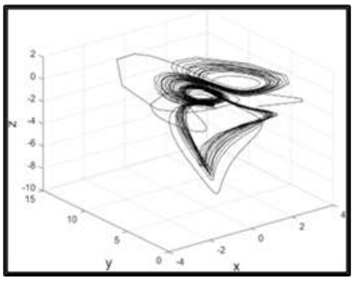


Figure 7. *2D phase plot of the attractors in (y -z) plane*

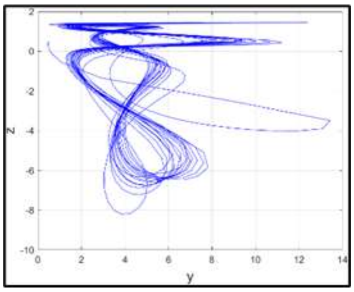


Figure 8. *3D phase plot of the attractors in*

It appears from fiigures (5-7) and figure (8) that the attractors of system (1) exhibts coplexare behaviors of chaotic dynamics.

3.6 Multistability

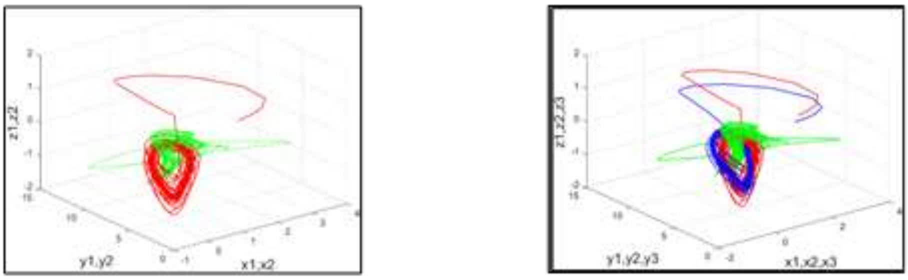
Attractors Coexisting Multistability in dynamical systems is the coexistence of two or more attractors with distinct initial circumstances but the same set of parameters.
Multiple in a Multistable System allows flexibility in system performance rich without changing parameters that lead to novel behavior figures 9(i) and l0(i) shows the coexistence of two attractors with different initial conditions and same, set of parameters, while figures 9(ii) and 10(ii) shows the coexistence of three attractors with different initial conditions and same set of parameters, values as given in table (3)

3.7. Bifurcation Analysis

Bifurcation is a useful manner to analyze the behavior of attractors of system (1), bifurcation diagram is a tool used in nonlinear theory to

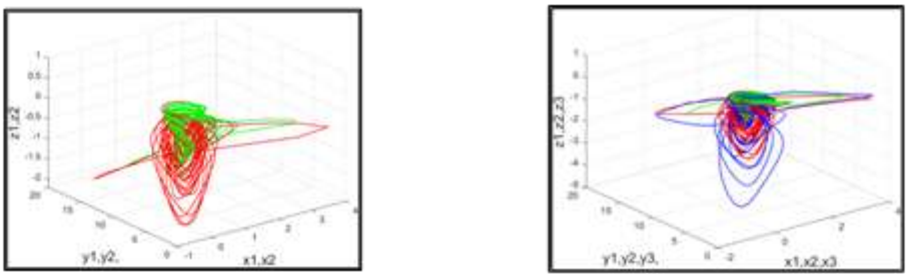
Table 3. *Coexistence with same parameter set and different initial conditions*

Initial values IC	Parameters	Color	Figures
$x_1 = 0.5, y_1 = 0.5, z_1 = 0.5$	a=4, b=0.1, c=1, $\sigma = 1$, $\beta = 0.05$, $, \alpha = 0.3$	Red Green Blue	Figure (9) (i), (ii)
$x_2 = 1.15, y_2 = 1.15, z_2 = 1.15$			
$x_3 = 1, y_2 = 1, z_3 = 1$			
$x_1 = 0.5, y_1 = 0.5, z_1 = 0.5$	a=4, b=0.1, c=1, $\sigma = 1$, $\beta = 0.05$, $, \alpha = 0.3$	Red Green Blue	Figure (10) (i), (ii)
$x_2 = 0.1, y_2 = 0.2, z_2 = 0.5$			
$x_3 = 0.3, y_2 = 0.2, z_3 = 0.5$			



(i) coexistence of two attractors (ii) coexistence of three attractors

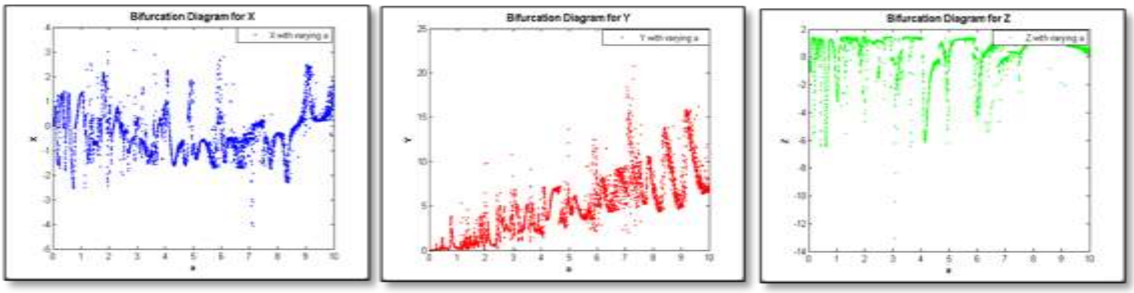
Figure 9. Coexistence of attractors of the hyper chaotic system (1) in (x, y, z) Space



(i) coexistence of two attractors (ii) coexistence of three attractors

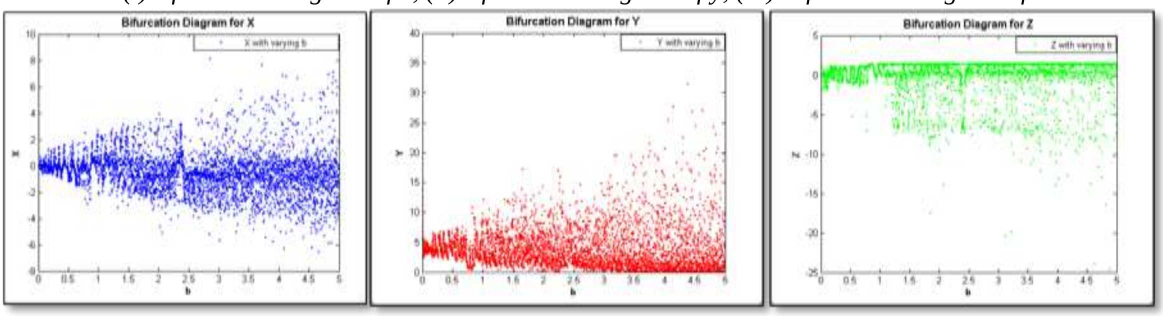
Figure 10. *coexistence of attractors of hyperchaotic System (1) in in (x, y, z) space.*

understand the system's dynamic behaviors [27,28]. The bifurcation diagrams of state variables (x, y, z) in relation to the parameter a and fixed $[b = \sigma, c, \alpha, \beta] = [1,1.5,0.3,0.05]$,and in relation to parameter b and fixed $[a = \sigma, c, \alpha, \beta] = [4,1,1.5,0.3,0.05]$, *are analysed in this section*, the numerical analysis started with initial conditions (IC) $[x_0, y_0, z_0] = [0.5,0.5,0.5]$ and $t_0 = 0, t_{step} = 0.5$ *and* $t_{end} = 5000$, be the starting (the attending time), step time. and finalization time (in second) respectively. The corresponding bifurcation diagram depicted in Figure (11) and Figure (12) all shows non-periodic dynamics, so the system (1) behaves chaotic.



(i) *(ii)* *(iii)*

Figure 11. *Bifurcation diagram of system (1) with respect to parameter a in interval* $a \in (0,10)$.
(i)Bifurcation diagram of x, (ii)Bifurcation diagram of y, (iii) Bifurcation diagram of z



(i) ***(ii)*** ***(iii)***

Figure (12): *Bifurcation diagram of system (1) with respect to parameter b in interval* $b \in [0,5]$.
(i)Bifurcation diagram of x, (ii)Bifurcation diagram of y, (iii) Bifurcation diagram of z

4. Adaptive Control and Synchronization Technique

In chaotic system there are many unstable orbits, and the chaotic attractor usually embedded with it an infinite number of unstable orbits. An unstable periodic or bit must be stabilized in order to Control chaos, by imposing only small stimulus to achieve the desired stability of the system. Adaptive Control and synchronization techniques are designed So system (1) is globally stabilized.

4.1 Adaptive Control Technique

To stabilize the hyperchaotic system (1), using adaptive Control technology, where the parameter (a) is unknown. Thus, the controlled hyperchaotic system

$$\dot{x} = x(a-y) + 0.3z + U_1(t)$$
$$\dot{y} = -y(1-x^2) + U_2(t) \dots\dots\dots\dots\dots\dots (10)$$
$$\dot{z} = -x(1.5-z) - 0.05z + U_3(t)$$

Where the constants U_1, U_2 *and* U_3 *a*re feedback controllers greater than zero to be designed to ensure (10) globally converges to zero, consider the functions of adaptive control.

$$U_1(t) = -x(\hat{a}-y) - 0.3z - \mu_1 x$$
$$U_2(t) = y(1-x^2) - \mu_2 y$$
$$U_3(t) = x(1.5-z) + 0.05z - \mu_3 z \dots\dots\dots\dots\dots (11)$$

Where $\hat{a}$ is the estimates parameter of a, and μ_i, (i=1,2,3) are positive substituting (11) into (10), we obtain:

$$\dot{x} = (a-\hat{a})\,x - \mu_1 x$$
$$\dot{y} = -\mu_2 y \dots\dots\dots\dots\dots (12)$$
$$\dot{z} = -\mu_3 z$$

let the parameter error estimation

$$e_a = a - \hat{a} \dots\dots\dots\dots\dots\dots\dots (13)$$

using (13) the closed-loop dynamic (12) can be expressed as

$$\dot{x} = e_a\, x - \mu_1 x$$
$$\dot{y} = -\mu_2 y \dots\dots\dots\dots\dots (14)$$
$$\dot{z} = -\mu_3 z$$

Let V the Lyapunov functions is positive definite on R^4

$$V(x,y,z,e_a) = \frac{1}{2}\,(x_2 + y^2 + z^2 + e^2{}_a) \dots\dots\dots\dots\dots\dots (15)$$

also

$$\dot{e}_a = -\hat{a} \dots\dots\dots\dots\dots\dots\dots\dots (16)$$

Differentiate (15) and substitute (12) and (13), we get

$$\dot{V} = -\mu_1 x^2 - \mu_2 y^2 - \mu_3 z^2 - e_a[\hat{a} + x^2] \dots\dots\dots\dots (17)$$

Assume that $\hat{a} = [\mu_4 e_a - x^2] \dots\dots\dots (18)$

Is the updated estimated parameter where $\mu_4 > 0$
Substitute (18) into (17), we get

$$\dot{V} = -\mu_1 x^2 - \mu_2 y^2 - \mu_3 z^2 - \mu_4 e_a^2 \dots\dots\dots\dots (19)$$

which is negative-definite on R^4 and the controller stability is ensured. So, the aforementioned proposition has been demonstrated.

Proposition (1):
For each initial value in equation (19) and estimated parameter provided by (20), and μ_1, μ_2, μ_3 *and* μ_4 are positive, the chaotic system (12) with unknown parameter is stabilized by adaptive control approach. This results in $V(x,y,z,e) < 0$

4.2 ADaptiv Synchronization.

In this section the adaptive synchronization technique of hyperchaotic system with unknown parameter a as drive (master), represented by:

$$\dot{x} = x(a-y) + 0.3z$$
$$\dot{y} = -y(1-x^2) \dots\dots\dots\dots\dots (20)$$
$$\dot{z} = -x(1.5-z) - 0.05z$$

While the slave (response) system considered as:

$$\dot{Y}_1 = ay_1 - y_1 y_2 + 0.3y_3 + U_1$$
$$\dot{Y}_2 = -y_2 + {y_1}^2 y_2 + U_2 \dots\dots\dots\dots\dots (21)$$
$$\dot{Y}_3 = -1.5y_1 - y_1 y_3 + 0.05y_3 + U_3$$

Where $y_1, y_2, y_3, y_4.$ are state variables and u_1, $u_2, u_3, u_4,$ are nonlinear controllers that need to be constructed to synchronize the two system (20) and (21).

The Synchronization error between two systems:

$$e_i = y_i - x_i\,, i = 1,2,3, \dots\dots\dots\dots\dots\dots (22)$$

using

$$\dot{e}_i = \dot{y}_i - \dot{x}_i$$

Substitute in (1) and (22), the following error dynamics easily obtained as

$$\dot{e}_1 = ae_1 - (e_1 e_2 + ye_1 + xe_2) + 0.3e_3 + u_1$$
$$\dot{e}_2 = -e_2 + (e_2 e_1^2 + ye_1 + xe_2) + u_2$$

$\dot{e_3} = -1.5e_1 + (e_1e_3 + ze_3 + ye_1) - 0.05e_3 + u_3$

u_1, u_2, u_3, u_4 are the adaptive control functions that are specified as:

$u_1 = -\hat{a}e_1 + (e_1e_2 + ye_1 + xe_2) + 0.3e_3 - \mu_1 e_1$
$u_2 = e_2 - (e_1^2 e_2 + ye_1 + xe_2) - \mu_2 e_2$
$u_3 = 1.5e_1 - (e_1e_3 + ze_3 + ye_1) + 0.05e_3 - \mu_3 e_3$

Where μ_1, μ_2 and μ_3 are positive real values and $\hat{a}$ is the estimate value of a.

Substitute (23) into (22) we get dynamical system of the synchronization error:
$\dot{e_1} = (a - \hat{a})e_1 - \mu_1 e_1$
$\dot{e_2} = -\mu_2 e_2$
$\dot{e_3} = -\mu_3 e_3$ .... (24)

Hence
$\dot{e_1} = e_a e_1 - \mu_1 e_1$
$\dot{e_2} = -\mu_2 e_2$
$\dot{e_3} = -\mu_3 e_3$ .... (25)

Where $e_a = a - \hat{a}$, $\dot{e_a} = -\dot{\hat{a}}$ ... ... (26)

The Lyapunov approach is used in order to prove the stability of the system (25).
Consider the quadratic Lyapunov quadratic function:
$v(e_1, e_2, e_3, e_a) = \frac{1}{2}(e_1^2, e_2^2, e_3^2, e_a^2)$ ... ...(27)
Which, on R^4 is positive definite.

After substituting the system (25) and (26), and differentiating equation (27) we obtain
$\dot{V} = - \mu_1 e_1^2 - \mu_2 e_2^2 - \mu_3 e_3^2 - e_a[\hat{a} - e_1^2]$................ (28)
The following law updates the estimated parameter
$\hat{a} = [e_1^2 + \mu_4 e_a]$............... (29)
where μ_4 is positive
Substitute (31) in (30), we get
$\dot{V} = -\mu_1 e_1^2 - \mu_2 e_2^2 - \mu_3 e_3^2 - \mu_4 e_4^2$
Which negative on R^4

So, based on Lyapunov stability. It is clear that the synchronization error and parameter error decay to zero in exponential with time for all initial condition, as shown in figure (13). Therefore, the following proposition is validated.

Proposition 2

The identical hyperchaotic system (22) and (23) with unknown parameter (a) are exponentially and globally synchronized for all initial conditions by adaptive control law (25), and parameter updating law (31) and $\mu_i, i = 1,2,3, \alpha$ are positive constant.

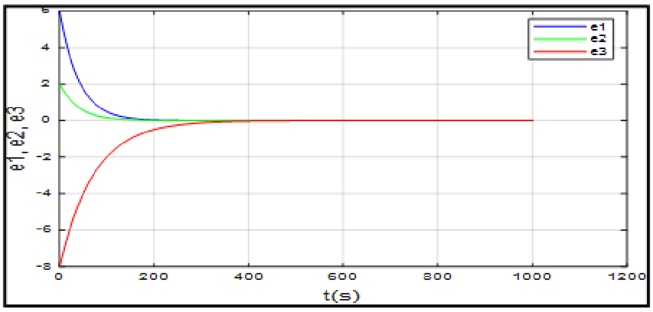


***Figure 13.** Convergent of trajectories for the dynamic of synchonizatio error*

5. Conclusion

In this paper, we introduce a novel three dimensional continuous-time autonomous dynamical system. A new method for constructing Lyapunov function using continued fractions was developed demonstrating consistency with established stability analysis techniques. Through a comprehensive investigation of the system dynamical properties - including equilibrium points, dissipativity, multi stability, time domain waveforms, phase portrait, bifurcations, and Lyapunov exponents we established it's hyperchaotic nature with a Kaplan-York dimension of D_{ky} =2.1621. A key finding is the system's multistability demonstrating the coexistence of attractors under distinct initial conditions additionally an adaptive control and synchronization strategy was successfully implemented, demonstrating effective stabilization of the hyper chaotic dynamics. These findings contribute to the ongoing study of hyperchaotic systems and their potential applications in secure communications, control theory, complex system modelling and nonlinear science.

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